Decay Properties and Asymptotic Profiles of Bounded Solutions to a Parabolic System of Chemotaxis in $\mathbb{R}^n$

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1 Introduction

In this paper, we study the large time behavior of bounded solutions to the Cauchy problem for the following system of partial differential equations in $\mathbb{R}^n$, $n \geq 2$:

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \nabla v), \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Delta v - v + u, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0, \quad v(x,0) = v_0, \qquad x \in \mathbb{R}^n. \tag{1.3}$$

The system (1.1), (1.2) is a mathematical model describing chemotaxis, that is, the directed movement of an organism in response to gradients of a chemical attractant(see [2, 10, 17]). $u(x,t)$ corresponds to the population of the organism at place x and time t, and $v(x,t)$ to the concentration of the chemical.

Throughout this paper, it is always assumed that

$$u_0,\ v_0,\ \partial_j v_0 \in L^1(\mathbb{R}^n) \cap \mathcal{B}(\mathbb{R}^n)\ (1 \le j \le n).$$

Here, we use the notation $\partial_j = \partial/\partial x_j$ for simplicity, and $\mathcal{B}(\mathbb{R}^n)$ is the Banach space of all bounded and uniformly continuous functions on $\mathbb{R}^n$ with the usual supremum norm. We write (1.1)–(1.3) in the form of the integral equation:

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u\nabla v)(s)ds, \tag{1.4}$$

$$v(t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-t+s}e^{(t-s)\Delta}u(s)ds, \tag{1.5}$$

where

$$(e^{t\Delta}f)(x) = \int_{\mathbb{R}^n} G(x-y,t)f(y)dy$$

and $G(x,t)$ is the heat kernel

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{|x|^2}{4t}).$$

A function $(u(x,t), v(x,t))$ on $\mathbb{R}^n \times [0,T]$ $(0 < T < \infty)$ is said to be a solution of (1.1)–(1.3) on $\mathbb{R}^n \times [0,T]$ if

$$u,\ v,\ \partial_j v \in C([0,T] : L^1(\mathbb{R}^n)) \cap C([0,T] : \mathcal{B}(\mathbb{R}^n)) \quad (1 \le j \le n)$$

and (u,v) satisfies (1.4), (1.5) on $[0,T]$. It is also said that (u,v) is a solution of (1.1)–(1.3) on $\mathbb{R}^n \times [0,\infty)$ if (u,v) is a solution of (1.1)–(1.3) on $\mathbb{R}^n \times [0,T]$ for every $0 < T < \infty$. Using standard regularity arguments for the heat equation(see Chapter IV of [11]), we see that (u,v) is a classical solution of (1.1)–(1.3), which satisfies

$$u, v \in C((0,T) : W^{2,p}(\mathbb{R}^n)) \cap C^1((0,T) : L^p(\mathbb{R}^n)) \quad \text{for every} \quad 1 < p < \infty.$$

It is shown in [14] that every bounded solution of (1.1)–(1.3) on $\mathbb{R}^2 \times [0,\infty)$ decays to zero as $t \to \infty$ and behaves like the heat kernel. We give the large time behavior for higher dimensional case in the following theorem. In what follows, $\|\cdot\|_p$ represents the usual L^p-norm.