Extension of a Geometric Stability Switch Criterion

Edoardo Beretta

Istituto di Biomatematica, Università di Urbino, I-61029 Urbino, Italy and

CIMAB: Centro Interuniversitario per la Matematica Applicata a Biologia, Medicina e Ambiente, Urbino, Italy

Yanbin Tang

Department of Mathematics, Huazhong University of Science and Technology

Wuhan, Hubei 430074, People's Republic of China

1 Introduction

In this paper we study the occurrence of any possible stability switch from the increase of the value of the time delay τ for general delay equation

$$x'(t) = F(x(t),\, x(t-\tau)) \tag{1.1}$$

where $x \in \Re^n$, $\tau \in \Re_{+0} = [0, +\infty)$ is a fixed delay and $F : C\left([-\tau, 0], \Re^n\right) \to \Re^n$ is of class C^1 with respect both $x(t)$ and $x(t-\tau)$.

We assume that any equilibrium x^* of (1.1) is delay dependent, i.e. $F(x_t) = 0$ gives a constant solution

$$x^* = x^*(\tau) \tag{1.2}$$

which is continuous and differentiable in τ.

The variation equation around x^* (set $u(t) = x(t) - x^*$)

$$\dot{u}(t) = \left(\frac{\partial F}{\partial x(t)}\right)_{x^*(\tau)} u(t) + \left(\frac{\partial F}{\partial x(t-\tau)}\right)_{x^*(\tau)} u(t-\tau) \tag{1.3}$$

gives the characteristic equation

$$\det\left\{\left(\frac{\partial F}{\partial x(t)}\right)_{x^*(\tau)} + e^{-\lambda\tau}\left(\frac{\partial F}{\partial x(t-\tau)}\right)_{x^*(\tau)} - \lambda I\right\} = 0 \tag{1.4}$$

which in general has delay dependent coefficients, where det denotes the determinant of a matrix, I is an identity matrix and λ are the corresponding characteristic roots.

We give the following definition:

Definition 1.1 A stability switch occurs at $\tau^* \in \Re_{+0}$ if crossing τ^* for increasing τ the stability of $x^*(\tau)$ changes from asymptotic stability to instability or vice versa.

The most general structure of characteristic equation (1.4) results to be

$$D(\lambda, \tau) = 0 \tag{1.5}$$

where

$$\begin{cases} D(\lambda,\tau) = P(\lambda,\tau) + \sum_{k=1}^{m} Q^{(k)}(\lambda,\tau)e^{-k\lambda\tau}, \\ P(\lambda,\tau) = \sum_{j=0}^{n} p_j(\tau)\lambda^j, \\ Q^{(k)}(\lambda,\tau) = \sum_{j=0}^{m_k} q_j^{(k)}(\tau)\lambda^j, k = 1,\ldots,m, \\ m, n, m_k \in N_0, \end{cases} \tag{1.6}$$

and $p_j(\tau)$, $q_j^{(k)}(\tau)$: $\Re_{+0} \to \Re$ are continuous and differentiable functions of $\tau \in \Re_{+0}$. Generally in (1.6) is $m \leq n$ but herefollowing we remove this condition.

We assume that $\lambda = 0$ cannot be a characteristic root, i.e.

$$D(0,\tau) \neq 0 \qquad \forall \tau \in \Re_{+0}. \tag{1.7}$$

In the study of the occurrence of stability switches the following is an essential result. Assume that we rewrite (1.5) as

$$D(\lambda,\tau) = \lambda^n + g(\lambda,\tau). \tag{1.8}$$

Then, the following theorem holds (see Freedman and Kuang [5]):

Theorem 1.1 *Assume that $g(\lambda,\tau)$ in (1.8) is an analytic function in λ and continuous in τ such that*

$$\alpha = \limsup_{\Re e \lambda \geq 0, |\lambda| \to \infty} \left| \lambda^{-n} g(\lambda,\tau) \right| < 1. \tag{1.9}$$

Then, as τ varies in $\Re_{+0}$,the sum of multiplicities of roots of $D(\lambda,\tau) = 0$ in the open right half-plane can change only if a root appears on or crosses the imaginary axis.

It is to be noticed that if in (1.6)

$$m_k < n, k = 1,\ldots,n, \tag{1.10}$$

i.e. the degree of polynomial $Q^{(k)}$ in λ is lower than the degree n polynomial P in λ, then assumption (1.9) holds true and Theorem 1.1 applies to the characteristic equation (1.5) and (1.6).

Rewrite $D(\lambda,\tau)$ in (1.6) like

$$D(\lambda,\tau) = p_n(\tau)\lambda^n + \left[\sum_{j=0}^{n-1} p_j(\tau)\lambda^j + \sum_{k=1}^{m} \left(\sum_{j=0}^{m_k} q_j^{(k)}(\tau)\lambda^j \right) e^{-k\lambda\tau} \right]$$

without loss of generality we assume $p_n(\tau) \equiv 1$ and define

$$g(\lambda,\tau) = \sum_{j=0}^{n-1} p_j(\tau)\lambda^j + \sum_{k=1}^{m} \left(\sum_{j=0}^{m_k} q_j^{(k)}(\tau)\lambda^j \right) e^{-k\lambda\tau}. \tag{1.11}$$