Commuting differential operators of type B_2

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1 Introduction

1.1. Several integral systems are accidentally related to root systems. Olshanetsky-Perelomov ([OP1], [OP2]) considered integrable n-particle models in dimension one arising from root systems. The systems of differential operators satisfied by zonal spherical functions give such integrable systems and these were generalized by Sekiguchi and Heckman-Opdam ([Sj], [HO]).

In [OOS] we announce a classification of integrable systems invariant under simple classical Weyl groups. The precise discussion has already been given by [OS] and [O] except for the case of type B_2. As is shown in [OS], the classification problem for type B_2 is reduced to a functional differential equation (1.4).

In §2 we give a complete list of solutions of this functional equation. Some solutions have already been obtained, after [OP2], by Inozemtsev [IM], [I] (See also [P]). The main result of §2 is Theorem 2.9, which is stated in §1.3 in a different form.

In §3 we examine the reducibility of the system obtained in §2. We note that if the system coincides with the system satisfied by zonal spherical functions of a semisimple Lie group, the reducibility is related to degenerate series representations.

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1.2. Now we give a quick review of the results in [OS, §6] concerning with type B_2. Let $W(B_2)$ be the Weyl group of type B_2, which is identified with the group of coordinate transformations of (x_1, x_2) generated by

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$(x_1,x_2)\mapsto(x_2,x_1)$ and $(x_1,x_2)\mapsto(x_1,-x_2)$. Consider $W(B_2)$-invariant differential operators

$$
(1.1)\qquad \begin{cases} P_1 = \partial_1^2+\partial_2^2+R(x), \\ P_2 = \partial_1^2\partial_2^2 + \text{lower order terms} \end{cases}
$$

which satisfies $[P_1,P_2]=0$ and ${}^tP_2=P_2$. Here we denote $\partial_1=\frac{\partial}{\partial x_1}$ and $\partial_2=\frac{\partial}{\partial x_2}$ for simplicity and the map ^{t} is the anti-automorphism of the algebra of differential operators such that ${}^ta(x)=a(x)$ for functions $a(x)$ and ${}^t\partial_i=-\partial_i$ for $i=1$ and 2. We assume that the coefficients of differential operators are extended to holomorphic functions on a Zariski open subset of an open connected neighborhood of the origin of the complexification $\mathbb{C}^2$ of $\mathbb{R}^2$.

The operators are proved to be expressed by even functions u and v of one variable as follows ([OS, Proposition 6.3]):

$$
(1.2)\qquad \begin{cases} P_1 &= \partial_1^2+\partial_2^2+u(x_1+x_2)+u(x_1-x_2)+v(x_1)+v(x_2), \\ P_2 &= \Big(\partial_1\partial_2+\dfrac{u(x_1+x_2)-u(x_1-x_2)}{2}\Big)^2+v(x_2)\partial_1^2+v(x_1)\partial_2^2 \\ & \quad +v(x_1)v(x_2)+T(x_1,x_2), \end{cases}
$$

where T is determined by the following equations up to a constant.

(1.3)

$$
\begin{cases} \partial_2 T = \frac{1}{2}v'(x_1)\big(u(x_1+x_2)-u(x_1-x_2)\big)+v(x_1)\big(u'(x_1+x_2)-u'(x_1-x_2)\big), \\ \partial_1 T = \frac{1}{2}v'(x_2)\big(u(x_1+x_2)-u(x_1-x_2)\big)+v(x_2)\big(u'(x_1+x_2)+u'(x_1-x_2)\big). \end{cases}
$$

As the compatibility condition for the existence of the solution T of the equation (1.3), we have an equation

(1.4)

$$
\begin{aligned} &\partial_2\Big(v'(x_2)\big(u(x_1+x_2)-u(x_1-x_2)\big)+2v(x_2)\big(u'(x_1+x_2)+u'(x_1-x_2)\big)\Big) \\ &=\partial_1\Big(v'(x_1)\big(u(x_1+x_2)-u(x_1-x_2)\big)+2v(x_1)\big(u'(x_1+x_2)-u'(x_1-x_2)\big)\Big), \end{aligned}
$$

which have been posed in [OS, Proposition 6.3] (cf. [P, §2.2.C]).

Conversely for any solution (u,v) of (1.4) and the pair (P_1,P_2) of the operators which are given by (1.2) with

$$
(1.5)\qquad T=\frac{1}{2}\Big(\partial_1^2-\partial_2^2\Big)\Big(V(x_1)\big(U(x_1+x_2)+U(x_1-x_2)\big)-G(x_1)\Big)
$$

under the notation in Remark 2.1 and Lemma 2.2, we have $[P_1,P_2]=0$.