Lower Estimates for the Growth of Painlevé Transcendents

By

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1 Introduction

Consider the first and the second Painlevé equations

(I) $$w'' = 6w^2 + z,$$

$(\mathrm{II})_\alpha$ $$w'' = 2w^3 + zw + \alpha, \quad \alpha \in \boldsymbol{C}$$

($' = d/dz$). All the solutions of these equations are meromorphic in the whole complex plane $\boldsymbol{C}$ ([5], [9]). Every solution of (I) is transcendental, and equation $(\mathrm{II})_\alpha$ admits a rational solution if and only if $\alpha \in \boldsymbol{Z}$ (e.g. [2], [8]); these equations define Painlevé transcendents.

The growth of a meromorphic function $f(z)$ is measured by the characteristic function defined by

$$T(r, f) = m(r, f) + N(r, f)$$

with

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \qquad \log^+ x = \max\{\log x, 0\},$$

$$N(r, f) = \int_0^r \big(n(t, f) - n(0, f)\big)\frac{dt}{t} + n(0, f)\log r;$$

here $n(r, f)$ denotes the number of poles in $|z| \le r$, each counted according to its multiplicity (for the notation of value distribution theory and basic facts, see [4], [6]). Also we use the notation $g(r) \ll h(r)$ if $g(r) = O(h(r))$ as $r \to \infty$.

The growth of each Painlevé transcendent is estimated as follows ([10], [11]):

Theorem A. *Let $w(z)$ be an arbitrary solution of* (I) *(resp.* $(\mathrm{II})_\alpha$*). Then, $T(r, w) \ll r^{5/2}$ (resp. $T(r, w) \ll r^3$).*

On the other hand, Mues and Redheffer [7] have shown the following:

Theorem B. *For every solution $w(z)$ of* (I)*, we have $\sigma(w) \ge 5/2$, where $\sigma(w) = \limsup_{r\to\infty} \log T(r, w)/\log r$.*

By these results, the order of the first Painlevé transcendents is 5/2.

In this paper we improve on the result of Theorem B, and under a certain condition, we give a lower estimate for $\sigma(w)$ of the second Painlevé transcendents. Our results are stated as follows:

Theorem 1.1. *For every solution $w(z)$ of* (I), *we have*

$$r^{5/2}/\log r \ll T(r,w) \ll r^{5/2}.$$

An arbitrary solution of (I) is expressible in the form $w(z) = -(u'(z)/u(z))'$, where $u(z)$ is an entire function called a *τ-function.* Note that it is uniquely determined apart from the factor $\exp(a_0 z + a_1)$ $(a_0, a_1 \in \boldsymbol{C})$.

Theorem 1.2. *For every solution of* (I), *its τ-function $u(z)$ satisfies*

$$r^{5/2}/\log r \ll T(r,u) \ll r^{5/2}.$$

Theorem 1.3. *Suppose that $2\alpha \in \boldsymbol{Z}$. Then, for every transcendental solution $w(z)$ of* $(\mathrm{II})_\alpha$, *we have $3/2 \le \sigma(w) \le 3$.*

Remark 1.1. The implicit coefficients of the relation in Theorem 1.1 are estimated as follows:

$$\liminf_{r\to\infty} T(r,w)(r^{5/2}/\log r)^{-1} \ge 4\cdot 10^{-11} K_0^{-5}, \quad \limsup_{r\to\infty} T(r,w) r^{-5/2} \le 2K_0/5,$$

where $K_0 = 1 + \limsup_{r\to\infty} n(r,w) r^{-5/2}$ $(< \infty)$.

Remark 1.2. For every solution $w(z)$ of (I), Boutroux [1] asserts the inequality $n(r,w) \gg r^{5/2}/\log r$, but his proof contains an incorrect part.

Remark 1.3. If $\alpha - 1/2 \in \boldsymbol{Z}$, then equation $(\mathrm{II})_\alpha$ admits a one-parameter family of solutions $\{v_c(z)\}_{c\in \boldsymbol{C}\cup\{\infty\}}$ such that $\sigma(v_c) = 3/2$ (see Section 4.3 and [2]).

2 Preliminaries

In this section, κ denotes an arbitrary positive numbere such that $\kappa > 1$. Let $\{c_j\}_{j=1}^{\infty}$ be a sequence satisfying $|c_1| \le |c_2| \le \cdots \le |c_j| \le \cdots$, where c_j $(j = 1, 2, ...)$ are not necessarily distinct. Now consider the summations

$$S_0(\{c_j\}, \kappa r, z) = \sum_{|c_j|<\kappa r} |z - c_j|^{-1}, \quad S_1(\{c_j\}, \kappa r, z) = \sum_{|c_j|<\kappa r} |z - c_j|^{-2}.$$

Put $\Delta_0(r) = \{z \mid |z| < r\}$. Let $\nu(r)$ denote the number of points c_j in $\Delta_0(r)$.

Lemma 2.1. *Suppose that $\nu(r) = O(r^\lambda)$ $(\lambda > 0)$. For every $r > r_0(\kappa)$, there exists a point $z_r \in \Delta_0(r) \setminus \Delta_0(r/\sqrt{2})$ with the properties*:

(2.1) $$S_0(\{c_j\}, \kappa r, z_r) \le 32(\kappa+1)\nu(\kappa r) r^{-1};$$

(2.2) $$S_1(\{c_j\}, \kappa r, z_r) \le 6\lambda_0 \nu(\kappa r) r^{-2} \log r.$$

Here $r_0(\kappa)$ is a sufficiently large positive number, and $\lambda_0 = \max\{1, \lambda/2\}$.