Quadratic Relations for Generalized Hypergeometric Functions ${}_pF_{p-1}$

Katsuyoshi Ohara, Yuichi Sugiki, and Nobuki Takayama

February 22, 2002, Revised September 27, 2002

1 Introduction

Let ${}_pF_{p-1}(a_1, \ldots, a_p, b_2, \ldots, b_p; z)$ be the generalized hypergeometric function

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_2)_n \cdots (b_p)_n} z^n, \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Theorem 1.1 *The generalized hypergeometric function ${}_pF_{p-1}$ satisfies the following quadratic relation*

$$(1) \qquad \sum_{i=1,j=1}^{p} \left(\theta^{i-1}\, {}_pF_{p-1}(A, B; z)\right) \frac{c_{ij}}{c_{11}} \left(\theta^{j-1}\, {}_pF_{p-1}(-A, 2-B; z)\right) = 1$$

for generic values of parameters a_i and b_j where $\theta = zd/dz$, $A = (a_1, \ldots, a_p)$, $B = (b_2, \ldots, b_p)$, $-A = (-a_1, \ldots, -a_p)$, $2 - B = (2 - b_2, \ldots, 2 - b_p)$. The number c_{ij} is the (i, j)-element of the transposed inverse of the intersection matrix of cocycles associated to ${}_pF_{p-1}$ and the intersection matrix is inductively determined with respect to p by the formula given in Theorem 8.2. For example, these relations for $p = 2, 3$ are as follows.

(a) $p = 2$

$$\begin{aligned}
& {}_2F_1(a_1, a_2, b_2; z)\, {}_2F_1(-a_1, -a_2, 2 - b_2; z) \\
+ \; & \frac{z}{e_2}\, {}_2F_1'(a_1, a_2, b_2; z)\, {}_2F_1(-a_1, -a_2, 2 - b_2; z) \\
- \; & \frac{z}{e_2}\, {}_2F_1(a_1, a_2, b_2; z)\, {}_2F_1'(-a_1, -a_2, 2 - b_2; z) \\
- \; & \frac{a_1 + a_2 - e_2}{a_1 a_2 e_2} z^2\, {}_2F_1'(a_1, a_2, b_2; z)\, {}_2F_1'(-a_1, -a_2, 2 - b_2; z) = 1
\end{aligned}$$

where $e_2 = b_2 - 1$ *and* $a_1 a_2 \neq 0, e_2, \notin \mathbf{Z}$.
(b) $p = 3$

$$\begin{aligned}
& {}_3F_2(A,B;z)\,{}_3F_2(-A,2-B;z) \\
+ & \frac{(-t_1+1)z}{t_2}\,{}_3F_2(A,B;z)\,{}_3F_2'(-A,2-B;z) \\
+ & \frac{z^2}{t_2}\,{}_3F_2(A,B;z)\,{}_3F_2''(-A,2-B;z) \\
+ & \frac{(t_1+1)z}{t_2}\,{}_3F_2'(A,B;z)\,{}_3F_2(-A,2-B;z) \\
+ & \frac{((t_2+1)s_1 - t_1 s_2 - s_3 - t_1)z^2}{t_2 s_3}\,{}_3F_2'(A,B;z)\,{}_3F_2'(-A,2-B;z) \\
+ & \frac{(s_1+s_2-t_1-t_2)z^3}{t_2 s_3}\,{}_3F_2'(A,B;z)\,{}_3F_2''(-A,2-B;z) \\
+ & \frac{z^2}{t_2}\,{}_3F_2''(A,B;z)\,{}_3F_2(-A,2-B;z) \\
+ & \frac{(s_1-s_2-t_1+t_2)z^3}{t_2 s_3}\,{}_3F_2''(A,B;z)\,{}_3F_2'(-A,2-B;z) \\
+ & \frac{(s_1-t_1)z^4}{t_2 s_3}\,{}_3F_2''(A,B;z)\,{}_3F_2''(-A,2-B;z) \\
= & 1
\end{aligned}$$

where $s_1 = a_1 + a_2 + a_3$, $s_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, $s_3 = a_1 a_2 a_3$, $t_1 = e_2 + e_3$, $t_2 = e_2 e_3$, $e_2 = b_2 - 1, e_3 = b_3 - 1$, *and* $A = (a_1, a_2, a_3, a_4)$, $B = (b_2, b_3, b_4)$, $-A = (-a_1, -a_2, -a_3, -a_4)$, $2 - B = (2 - b_2, 2 - b_3, 2 - b_4)$. *Parameters must satisfy the condition* $a_1 a_2 a_3 \neq 0, b_2, b_3 \notin \mathbf{Z}$.

Some identities for hypergeometric functions have geometric meaning behind. Aomoto proposed a method to study hypergeometric functions as pairings of cycles and cocycles about 30 years ago [1]. This ingenious point of view has enabled us to yield a lot of formulas for hypergeometric functions.

The identities we have presented above are quadratic relations for ${}_pF_{p-1}$. We will see that it also has a geometric meaning based on a work of Cho and Matsumoto. They proved an analog of Riemann's period relation for intersection numbers of cycles and cocycles and associated period integral, of which entries are nothing but hypergeometric functions [2]. The period relation yields a quadratic relation of hypergeometric functions. Therefore, the