Evaluation of Stokes multipliers for a certain system of differential equations corresponding to a rigid local system

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0. Introduction.

In [BJL2] Balser, Jurkat and Lutz studied a system of linear differential equations of Birkhoff canonical form of Poincaré rank one, i.e.

$$\frac{dZ}{dx} = \left(\Lambda + \frac{A}{x}\right)Z, \tag{0.1}$$

where A is a constant $n \times n$ matrix and $\Lambda = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$. Under the assumuption that Λ has all distinct diagonal elements, they showed that the Stokes multipliers for (0.1) can be expressed by using the connection coefficients for the associated system of linear differential equations

$$(\Lambda - tI_n)\frac{dY}{dt} = (A + I_n)Y, \tag{0.2}$$

which is Fuchsian and has regular singular points $\lambda_1, \dots, \lambda_n, \infty$. Balser [Ba] studied the same problem for the system (0.1) with $\Lambda = \mathrm{diag}[0, \dots, 0, 1]$ whose associated system (0.2) is equivalent to the differential equation satisfied by the generalized hypergeometric series ${}_nF_{n-1}$, and evaluated the Stokes multipliers for (0.1) explicitly. As Balser's result suggests, we do not need the assumption that Λ has all distinct diagonal elements to establish the relation between the Stokes multipliers for (0.1) and the connection coefficients for (0.2) (Theorem 5.2). Then the essential part of his result is the evaluation of the connection coefficients for the associated system.

Recently we have shown that, if the monodromy representation of the system (0.2) defines a rigid local system (we call such a system (0.2) *rigid*), the solutions of the system have an integral representation ([Ha4]). Then, for rigid systems, by using the integral representation we can evaluate the connection coefficients and hence the Stokes multipliers for the corresponding system (0.1). (Note that Balser's system (0.2) is also rigid.) In this paper we take a rigid system of rank 4 whose solution can be represented by a double integral, evaluate the connection coefficients by using the integral, and evaluate the Stokes multipliers for the corresponding system (0.1). It seems hard to describe the connection coefficients for a general rigid system; however, the computation in this paper will be applied to each rigid system.

1. A Fuchsian system of rank 4.

Let λ_1, λ_2 be distinct complex numbers, and set $\Lambda = \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2, \lambda_2]$. We consider the system of linear differential equations

$$(tI_4 - \Lambda)\frac{dY}{dt} = AY \tag{1.1}$$

with a constant 4×4 matrix A. The system (1.1) is Fuchsian with regular singular points $\lambda_1, \lambda_2, \infty$. The residue matrices at $\lambda_1, \lambda_2, \infty$ are

$$A_1 := \begin{pmatrix} I_2 & \\ & O \end{pmatrix} A, \quad A_2 := \begin{pmatrix} O & \\ & I_2 \end{pmatrix} A, \quad \text{and} \quad -A$$

respectively. We assume that

$$A_1 \sim \mathrm{diag}[a_1, a_2, 0, 0], \; A_2 \sim \mathrm{diag}[0, 0, b_1, b_2], \; A \sim \mathrm{diag}[\mu_1, \mu_1, \mu_2, \mu_3], \tag{1.2}$$

where $a_1, a_2, b_1, b_2, \mu_1, \mu_2, \mu_3$ are complex numbers satisfying $a_1+a_2+b_1+b_2 = 2\mu_1+\mu_2+\mu_3$. Moreover we assume

$$a_j, b_j, \mu_\ell, a_1 - a_2, b_1 - b_2, \mu_\ell - \mu_m, a_j - \mu_1, b_j - \mu_1, a_j + b_k - \mu_1 - \mu_2 \notin \mathbf{Z} \tag{1.3}$$

for $j, k = 1, 2$ and $\ell, m = 1, 2, 3$ with $\ell \neq m$. Then it is shown that the system (1.1) is irreducible and rigid, and that the matrix A can be determined uniquely up to gauge transformations $Y \mapsto PY$ with $P \in \mathrm{GL}(4, \mathbf{C})$ ([ST],[Ha1]). This system first appeared in the classification of rigid systems ([O2], [Y], [ST]), and Mimachi [M] found an integral representation of its solutions. This integral representation is rediscovered in [Ha4] as an example of the general theory. In the sequel we fix the matrix A as

$$A = \begin{pmatrix} a_1 & 0 & a_{13} & a_{14} \\ 0 & a_2 & a_{23} & a_{24} \\ a_{31} & a_{32} & b_1 & 0 \\ a_{41} & a_{42} & 0 & b_2 \end{pmatrix}, \tag{1.4}$$

where

$$\begin{cases} a_{13} = \dfrac{(\mu_1 - a_1)(a_1 + b_2 - \mu_1 - \mu_2)}{a_1 - a_2}, & a_{14} = \dfrac{(\mu_1 - a_1)(a_1 + b_1 - \mu_1 - \mu_2)}{a_1 - a_2}, \\ a_{23} = \dfrac{(\mu_1 - a_2)(a_2 + b_2 - \mu_1 - \mu_2)}{a_2 - a_1}, & a_{24} = \dfrac{(\mu_1 - a_2)(a_2 + b_1 - \mu_1 - \mu_2)}{a_2 - a_1}, \\ a_{31} = \dfrac{(\mu_1 - b_1)(a_2 + b_1 - \mu_1 - \mu_2)}{b_1 - b_2}, & a_{32} = \dfrac{(\mu_1 - b_1)(a_1 + b_1 - \mu_1 - \mu_2)}{b_1 - b_2}, \\ a_{41} = \dfrac{(\mu_1 - b_2)(a_2 + b_2 - \mu_1 - \mu_2)}{b_2 - b_1}, & a_{42} = \dfrac{(\mu_1 - b_2)(a_1 + b_2 - \mu_1 - \mu_2)}{b_2 - b_1} \end{cases} \tag{1.5}$$

([Ha1],[Ha4]). Then we have

Theorem 1.1. ([Ha4, Proposition 5.10]) *Every solution of (1.1) can be represented by the integral*

$$Y(t) = \int_\Delta \Phi U d\tau_1 \wedge d\tau_2, \tag{1.6}$$