Weyl Group Symmetry of Type $D_5^{(1)}$ in the $q-$Painlevé V Equation

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1 Introduction

Since the singularity confinement criterion was introduced as a discrete analogue of the Painlevé test [2], many discrete analogues of Painlevé equations have been proposed and extensively studied [3, 10]. Discrete Painlevé equations have been considered as 2-dimensional non-autonomous birational dynamical systems which satisfy this criterion and which have limiting procedures to the (continuous) Painlevé equations. In recent years it was shown by Sakai that all of these (from the point of view of symmetries) are obtained by studying rational surfaces in connection with extended affine Weyl groups [11].

On the other hand, recently Kajiwara *et al* (KNY) [6] have proposed a birational representation of the extended Weyl groups $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ on the field of rational functions $\mathbb{C}(x_{ij})$, which is expected to provide higher order discrete Painlevé equations (however, this representation is not always faithful, for example it is not faithful in the case where m or n equals 1 and in the case of $m = n = 2$). In the case of $m = 2$ and $n = 3, 4$, the actions of the translations can be considered to be 2-dimensional non-autonomous discrete dynamical systems and therefore to correspond to discrete Painlevé equations. Special solutions and some properties of these equations have been studied by several authors [5, 8]. In the case of $m = 2$ and $n = 4$, the action of the translation was thought to be a symmetric form of the q-discrete analogue of Painlevé V equation (q-P_V). However, the symmetry $\widetilde{W}(A_1^{(1)} \times A_3^{(1)})$ does not coincides with any symmetry of discrete Painlevé equations in Sakai's list, (in the case of $m = 2$ and $n = 3$, it coincides with an equation, which is associated with a family of $A_3^{(1)}$ surfaces and whose symmetry is $\widetilde{W}(A_1^{(1)} \times A_2^{(1)})$, in Sakai's list). So it is natural to suspect that the symmetry might be a subgroup of a larger group associated with some family of rational surfaces.

In this paper we show that in the case of $m = 2$ and $n = 4$ the action of the translation can be lifted to an automorphism of a family of rational surfaces of the type $A_3^{(1)}$, i.e. surfaces such that the type of the configuration of irreducible components of their anti-canonical divisors is $A_3^{(1)}$, and therefore that the group of these automorphisms is $\widetilde{W}(D_5^{(1)})$ (hence it is not q-P_V by Sakai's classification). The action can be decomposed into two mappings which are conjugate to the q-P_{VI} equation. It is also shown that the subgroup of automorphisms which commute with the original translation is isomorphic to $\mathbb{Z} \times \widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$.

2 Birational representation of $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$

The birational representation of $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ on $\mathbb{C}(x_{i,j})$ proposed by Kajiwara *et al* (KNY) [6] is an action on $\mathbb{C}(x_{i,j})$ ($i = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, n-1$ and the indices i, j are considered in modulo $m\mathbb{Z}, n\mathbb{Z}$ respectively) defined as follows.

We write the elements of the Weyl group corresponding to the simple roots as

$$r_i \in W(A_{m-1}^{(1)}),\ s_j \in W(A_{n-1}^{(1)})$$

and the elements corresponding to the rotations of the Dynkin diagrams as

$$\pi \in \mathrm{Aut}(\mathrm{Dynkin}(A_{m-1}^{(1)})),\ \rho \in \mathrm{Aut}(\mathrm{Dynkin}(A_{n-1}^{(1)})).$$

The action of these elements on $\mathbb{C}(x_{i,j})$ are defined as

$$r_i(x_{ij}) = x_{i+1,j}\frac{P_{i,j-1}}{P_{ij}}, \quad r_i(x_{i+1,j}) = x_{ij}\frac{P_{ij}}{P_{i,j-1}}, \quad r_k(x_{ij}) = x_{ij}, \quad (k \neq i, i-1),$$

$$s_j(x_{ij}) = x_{i,j+1}\frac{Q_{i-1,j}}{Q_{ij}}, \quad s_j(x_{i,j+1}) = x_{ij}\frac{Q_{ij}}{Q_{i-1,j}}, \quad s_k(x_{ij}) = x_{ij}, \quad (k \neq j, j-1),$$

$$\pi(x_{ij}) = x_{i+1,j}, \quad \rho(x_{i,j}) = x_{i,j+1},$$

where

$$P_{ij} = \sum_{a=0}^{n-1}\left(\prod_{k=0}^{a-1} x_{i,j+k+1} \prod_{k=a+1}^{n-1} x_{i+1,j+k+1}\right), \qquad Q_{ij} = \sum_{a=0}^{m-1}\left(\prod_{k=0}^{a-1} x_{i+k+1,j} \prod_{k=a+1}^{m-1} x_{i+k+1,j+1}\right).$$

For example in the $(m,n) = (2,4)$ case,

$$P_{00} = x_{1,2}x_{1,3}x_{1,0} + x_{0,1}x_{1,3}x_{1,0} + x_{0,1}x_{0,2}x_{1,0} + x_{0,1}x_{0,2}x_{0,3}$$

and $Q_{00} = x_{0,1} + x_{1,0}$. It was shown by KNY that this action is a representation of $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ as automorphisms of the field $\mathbb{C}(x_{i,j})$. But it is still an open problem when this representation is faithful. In the case of $m = 2$ and $n = 3, 4$, one can see it is faithful by considering the actions on the root systems which we discuss later.

In the case of $(m,n) = (2,4)$, the variable transformation:

$$\left(\frac{x_{0,j}x_{1,j}}{x_{0,j+1}x_{1,j+1}}\right)^{1/2} = a_j, \qquad \left(\frac{x_{0,j}x_{0,j+1}}{x_{1,j}x_{1,j+1}}\right)^{1/2} = f_j\ ,$$