On a class of algebraic solutions to the Painlevé VI equation, its determinant formula and coalescence cascade

Tetsu Masuda
Department of Mathematics, Kobe University,
Rokko, Kobe, 657-8501, Japan
masuda@math.kobe-u.ac.jp

Abstract

A determinant formula for a class of algebraic solutions to the Painlevé VI equation ($\mathrm{P_{VI}}$) is presented. This expression is regarded as a special case of the universal characters. The entries of the determinant are given by the Jacobi polynomials. Degeneration to the rational solutions of $\mathrm{P_V}$ and $\mathrm{P_{III}}$ is discussed, using the coalescence procedure. The relationship between Umemura polynomials associated with $\mathrm{P_{VI}}$ and our formula is also discussed.

1 Introduction

Enlarging the work by Yablonskii and Vorob'ev for $\mathrm{P_{II}}$ [28] and Okamoto for $\mathrm{P_{IV}}$ [23], Umemura has introduced special polynomials associated with a class of algebraic (or rational) solutions to each of the Painlevé equations $\mathrm{P_{III}}$, $\mathrm{P_V}$ and $\mathrm{P_{VI}}$ [27]. These polynomials are generated by the Toda equation that arises from the Bäcklund transformations of each Painlevé equation. It has been also found that the coefficients of the polynomials admit mysterious combinatorial properties [15, 26].

It is remarkable that some of these polynomials are expressed as a specialization of the Schur functions. Yablonskii-Vorob'ev polynomials are expressible by 2-core Schur functions, and Okamoto polynomials by 3-core Schur functions [7, 8, 16]. It is now recognized that these structures reflect the affine Weyl group symmetry, as groups of the Bäcklund transformations [29]. The determinant formulas of Jacobi-Trudi type for Umemura polynomials of $\mathrm{P_{III}}$ and $\mathrm{P_V}$ resemble each other. In both cases, they are expressed by 2-core Schur functions, and entries of the determinant are given by the Laguerre polynomials [5, 17].

Furthermore, in a recent work, it has been revealed that the entire families of the characteristic polynomials for rational solutions of $\mathrm{P_V}$, which include Umemura polynomials for $\mathrm{P_V}$ as a special case, admit more general structures [12]. Namely, they are expressed in terms of the universal characters that are a generalization of the Schur functions. The latter are the characters of the irreducible polynomial representations of $GL(n)$, while the former were introduced to describe the irreducible rational representations [11].

What kind of determinant structures do Umemura polynomials for $\mathrm{P_{VI}}$ admit? Recently, Kirillov and Taneda have introduced a generalization of Umemura polynomials for $\mathrm{P_{VI}}$ in the context of combinatorics and have shown that their polynomials degenerate to the special polynomials for $\mathrm{P_V}$ in some limit [9, 10]. This result suggests that the special polynomials associated with a class of algebraic solutions to $\mathrm{P_{VI}}$ are also expressible by the universal characters.

In this paper, we consider $\mathrm{P_{VI}}$

$$\begin{aligned}\frac{d^2y}{dt^2}=&\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\&+\frac{y(y-1)(y-t)}{2t^2(t-1)^2}\left[\kappa_\infty^2-\kappa_0^2\frac{t}{y^2}+\kappa_1^2\frac{t-1}{(y-1)^2}+(1-\theta^2)\frac{t(t-1)}{(y-t)^2}\right],\end{aligned}\tag{1.1}$$

where κ_∞, κ_0, κ_1 and θ are parameters. As is well known [21], P_{VI} (1.1) is equivalent to the Hamilton system

$$S_{VI}: \qquad q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad ' = t(t-1)\frac{d}{dt}, \tag{1.2}$$

with the Hamiltonian

$$\begin{gathered} H = q(q-1)(q-t)p^2 - [\kappa_0(q-1)(q-t) + \kappa_1 q(q-t) + (\theta-1)q(q-1)]p + \kappa(q-t), \\ \kappa = \frac{1}{4}(\kappa_0+\kappa_1+\theta-1)^2 - \frac{1}{4}\kappa_\infty^2. \end{gathered} \tag{1.3}$$

In fact, the equation for $y = q$ is nothing but P_{VI} (1.1).

The aim of this paper is to investigate a class of algebraic solutions to P_{VI} (or S_{VI}) that originate from the fixed points of the Bäcklund transformations corresponding to Dynkin automorphisms and, then to present its explicit determinant formula.

Let us remark on the terminology of "algebraic solutions". P_{VI} admits several classes of algebraic solutions [1, 2, 3, 13, 14], and the classification has not yet been established. In this paper, we concentrate our attention to the above restricted class of algebraic solutions.

This paper is organized as follows. In Section 2, we first present a determinant formula for a family of algebraic solutions to P_{VI} (or S_{VI}). This expression is also a specialization of the universal characters, and the entries of the determinant are given by the Jacobi polynomials. The symmetry of P_{VI} is described by the affine Weyl group of type $D_4^{(1)}$. In Section 3, as a preparation for constructing special solutions, we present a symmetric description of Bäcklund transformations for P_{VI} [6, 20]. We also derive several sets of bilinear equations for the τ-functions. In Section 4, starting from a seed solution on fixed points of a Dynkin automorphism, we construct a family of algebraic solutions to P_{VI} (or S_{VI}) by application of Bäcklund transformations. A family of special polynomials is extracted as the non-trivial factor of the τ-function, and our algebraic solutions are expressed by a ratio of these polynomials. A proof of our result is given in Section 5.

As is well known, P_{VI} degenerates to $P_V, \ldots, P_I$ by successive limiting procedures [25, 4]. In Section 6, we show that the family of algebraic solutions to P_{VI} given in Section 2 degenerate to rational solutions to P_V and P_{III} with the same determinant structures. Section 7 is devoted to discussing the relationship to the original Umemura polynomials for P_{VI}.

2 A determinant formula

Definition 2.1 *Let $p_k = p_k^{(c,d)}(x)$ and $q_k = q_k^{(c,d)}(x)$, $k \in \mathbb{Z}$, be two sets of polynomials defined by*

$$\begin{gathered} \sum_{k=0}^{\infty} p_k^{(c,d)}(x)\lambda^k = G(x;c,d;\lambda), \quad p_k^{(c,d)}(x) = 0 \text{ for } k<0, \\ q_k^{(c,d)}(x) = p_k^{(c,d)}(x^{-1}), \end{gathered} \tag{2.1}$$

respectively, where the generating function $G(x;c,d;\lambda)$ is given by

$$G(x;c,d;\lambda) = (1-\lambda)^{c-d}(1+x\lambda)^{-c}. \tag{2.2}$$

For $m,n \in \mathbb{Z}_{\geq 0}$, we define a family of polynomials $R_{m,n} = R_{m,n}(x;c,d)$ by

$$R_{m,n}(x;c,d) = \begin{vmatrix} q_1 & q_0 & \cdots & q_{-m+2} & q_{-m+1} & \cdots & q_{-m-n+3} & q_{-m-n+2} \\ q_3 & q_2 & \cdots & q_{-m+4} & q_{-m+3} & \cdots & q_{-m-n+5} & q_{-m-n+4} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1} & q_{2m-2} & \cdots & q_m & q_{m-1} & \cdots & q_{m-n+1} & q_{m-n} \\ p_{n-m} & p_{n-m+1} & \cdots & p_{n-1} & p_n & \cdots & p_{2n-2} & p_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4} & p_{-n-m+5} & \cdots & p_{-n+3} & p_{-n+4} & \cdots & p_2 & p_3 \\ p_{-n-m+2} & p_{-n-m+3} & \cdots & p_{-n+1} & p_{-n+2} & \cdots & p_0 & p_1 \end{vmatrix}. \tag{2.3}$$