Simultaneous effects of homogenization and vanishing viscosity in fully nonlinear elliptic equations*

Kazuo Horie[1)] and Hitoshi Ishii[2)]

1 Introduction

We consider the partial differential equation

$$(\mathrm{P})_\varepsilon \qquad F(x, x/\varepsilon, u^\varepsilon(x), Du^\varepsilon(x), \delta D^2 u^\varepsilon(x)) = 0 \qquad \text{in } \mathbf{R}^n,$$

where ε and $\delta \equiv \delta(\varepsilon)$ are two positive parameters, $F \in C(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n)$, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices, u^ε is the unknown, and Du^ε and D^2u^ε denote the gradient and Hessian of u^ε, respectively. The parameter δ will be given as a function of ε, that is, $\delta = \delta(\varepsilon)$. A typical example of $\delta(\varepsilon)$ is: $\delta(\varepsilon) = \varepsilon^a$, where $0 \le a < \infty$. We always assume

(A1) F is uniformly elliptic, that is, there are constants $0 < \theta \le \Theta$ for which if $X, Y \in \mathcal{S}^n$ and $Y \ge 0$, then

$$F(x, y, u, p, X) - \Theta \operatorname{tr} Y \le F(x, y, u, p, X + Y) \le F(x, y, u, p, X) - \theta \operatorname{tr} Y;$$

(A2) the function: $y \mapsto F(x, y, u, p, X)$ is periodic with period $\mathbf{Z}^n$, that is,

$$F(x, y + z, u, p, X) = F(x, y, u, p, X) \qquad \text{for all } z \in \mathbf{Z}^n;$$

and

(A3) there is a constant $\lambda > 0$ such that the function: $u \mapsto F(x, y, u, p, X) - \lambda u$ is non-decreasing in $\mathbf{R}$ for any $(x, y, p, X) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathcal{S}^n$.

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* Major part of this paper was prepared by the first author and delivered in a meeting held at Tokyo Metropolitan University on August, 30, 1999.

1) Died on October 2, 1999, when he was a graduate student of Tokyo Metropolitan University.

2) Supported in part by Grant-in-Aid for Scientific Research, No. 12440044, JSPS.

We investigate the asymptotic behavior of the solution u^ε of $(\mathrm{P})_\varepsilon$ as $\varepsilon \to 0$. The parameters ε and δ represent a state in the processes of homogenization and vanishing viscosity in $(\mathrm{P})_\varepsilon$, respectively, and our motivation to studying $(\mathrm{P})_\varepsilon$ is to understand the simultaneous effects of the periodic homogenization and vanishing viscosity in $(\mathrm{P})_\varepsilon$.

Let us describe briefly our results on the effects of the homogenization and vanishing viscosity. For this, we formulate the following three cell problems. Henceforth $C(\mathbf{R}^n/\mathbf{Z}^n)$ denotes the space of periodic functions u on $\mathbf{R}^n$ with period $\mathbf{Z}^n$. Fix $(\hat{x},\hat{u},\hat{p},\hat{X}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$. The first cell problem is to find a pair of $\mu \in \mathbf{R}$ and $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a (viscosity) solution of

$$(\mathrm{CP})_2 \qquad F(\hat{x},y,\hat{u},\hat{p},\hat{X}+D^2v(y)) = \mu \qquad \text{in } \mathbf{R}^n.$$

The second cell problem is to find a pair $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a (viscosity) solution of

$$(\mathrm{CP})_{12} \qquad F(\hat{x},y,\hat{u},\hat{p}+Dv(y),D^2v(y)) = \mu \qquad \text{in } \mathbf{R}^n.$$

The third cell problem is to find a pair $(\mu, v) \in \mathbf{R}\times C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a (viscosity) solution of

$$(\mathrm{CP})_1 \qquad F(\hat{x},y,\hat{u},\hat{p}+Dv(y),0) = \mu \qquad \text{in } \mathbf{R}^n.$$

In this paper we deal with fully nonlinear PDE which may be degenerate elliptic and which may not have classical solutions, and we adapt the notion of viscosity solution (see [CIL2]). Henceforth we suppress the word "viscosity" and, for instance, we call a viscosity solution simply a solution.

Under appropriate hypotheses each of these problems $(\mathrm{CP})_2$, $(\mathrm{CP})_{12}$, and $(\mathrm{CP})_1$ has a solution (μ, v) and moreover, the value of μ is determined uniquely while the function v is not determined uniquely. The correspondence of $(\hat{x},\hat{u},\hat{p},\hat{X})$ to this value μ is called the homogenized or effective function and denoted by $\bar{F}_2(\hat{x},\hat{u},\hat{p},\hat{X})$, $\bar{F}_{12}(\hat{x},\hat{u},\hat{p})$, and $\bar{F}_1(\hat{x},\hat{u},\hat{p})$, respectively, in problems $(\mathrm{CP})_2$, $(\mathrm{CP})_{12}$, and $(\mathrm{CP})_1$.

Four cases arise in our study of the asymptotics for $(\mathrm{P})_\varepsilon$. Case 1: $\lim_{\varepsilon\to 0}\delta(\varepsilon) \in (0,\infty)$. Case 2: $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = \infty$. Case 3: $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon \in (0,\infty)$. Case 4: $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = 0$. We may assume by a simple normalization that $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 1$ in Case 1 and $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = 1$ in Case 3.

Our main results state that under appropriate hypotheses, the solutions u^ε of $(\mathrm{P})_\varepsilon$ converge uniformly on $\mathbf{R}^n$, to the solution $u \in \mathrm{BUC}(\mathbf{R}^n)$ of

$$\bar{F}_2(x,u(x),Du(x),D^2u(x)) = 0 \qquad \text{in } \mathbf{R}^n,$$

$$\bar{F}_2(x,u(x),Du(x),0) = 0 \qquad \text{in } \mathbf{R}^n,$$