STABILITY OF CONSTANT EQUILIBRIUM FOR THE MAXWELL-HIGGS EQUATIONS

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Dedicated to Professor T. Nishida
on his 60th birthday

§1. Introduction and theorem

We consider the Maxwell-Higgs equations in space time dimensions 2 + 1 and 3 + 1:

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu, \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.1}$$

$$D_\mu D^\mu \varphi = \frac{1}{2}\big(m^2\varphi - \frac{2m^2}{M^2}|\varphi|^2\varphi\big), \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.2}$$

where $n = 2$ or 3, M and m are positive constants, A^μ are real-valued functions, φ is a complex-valued function and

$$j^\mu = -i\big\{\varphi\overline{(D^\mu\varphi)} - (D^\mu\varphi)\overline{\varphi}\big\},$$
$$D^\mu = \partial^\mu + iA^\mu.$$

Here and hereafter, we follow the convention that Greek indices take values in $\{0, 1, \cdots, n\}$ while Latin indices are valued in $\{1, \cdots, n\}$. Indices repeated lower and upper are summed. The space $\mathbf{R}^{n+1}$ is the $n+1$ dimensional Euclidean space equipped with the flat Minkowski metric

$$(g_{\alpha\beta}) = diag(1, -1, \cdots, -1).$$

Indices are raised and lowered using the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. We put $x^0 = t$ and $\partial_\alpha = \partial/\partial x^\alpha$.

The potential $V(|\varphi|) = \frac{m^2}{4M^2}(M^2/2 - |\varphi|^2)^2$ associated with the right hand side of (1.2) is called the Higgs potential and it has equilibria 0 and $z = Me^{i\theta}/\sqrt{2}$, $\theta \in \mathbf{R}$. The latter equilibria correspond to the degenerate ground state, which is

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the vacuum with vacuum expectation value $M/\sqrt{2}$. All equilibria $z = Me^{i\theta}/\sqrt{2}$, $\theta \in \mathbf{R}$ are equivalent from a physical point of view, because this system has $U(1)$ symmetry. If one of them is spontaneously chosen, for example, if $z = M/\sqrt{2}$ is chosen as a vacuum equilibrium, then the so-called spontaneous breakdown of symmetry will happen and the photon described by A^μ will have mass M. This is called the Higgs mechanism.

We look at this mechanism in more details (see [1, Section 13.3]). We first note that equations (1.1)-(1.2) are invariant under the following gauge transformation:

$$\tilde{A}^\mu = A^\mu - \partial^\mu \chi,$$
$$\tilde{\varphi} = e^{-i\chi}\varphi,$$

where $\chi(t,x)$ is an arbitrary smooth real-valued function. If φ is close to the vacuum equilibrium $M/\sqrt{2}$, then we can take $\chi = \arg\varphi$, $-\pi < \arg\varphi \le \pi$. In that case, both $\tilde{A}^\mu$ and $\tilde{\varphi}$ become real-valued functions. Such choice of gauge is called the unitary gauge. Again we denote $\tilde{A}^\mu$ and $\tilde{\varphi}$ by A^μ and φ, respectively. We put $\varphi = \phi + M/\sqrt{2}$ and suppose that the fluctuation (A^μ, ϕ) from the vacuum $(0,0)$ is small in the new dynamical variables. Then, the Cauchy problem for (1.1)-(1.2) is reduced to the following:

$$(1.3) \qquad (\Box + M^2)A^\mu - \partial^\mu\partial_\nu A^\nu = -2\sqrt{2}MA^\mu\phi - 2A^\mu\phi^2, \qquad (t,x) \in \mathbf{R}^{1+n},$$

$$(1.4) \qquad (\Box + m^2)\phi = \frac{M}{\sqrt{2}}A_\nu A^\nu + A_\nu A^\nu \phi - \frac{3m^2}{\sqrt{2}M}\phi^2 - \frac{m^2}{M^2}\phi^3, \qquad (t,x) \in \mathbf{R}^{1+n},$$

$$(1.5) \qquad (A^\mu(0), \partial_t A^\mu(0)) = (\alpha^\mu, \beta^\mu), \quad (\phi(0), \partial_t\phi(0)) = (\phi_0, \phi_1),$$

$$(1.6) \qquad \phi\partial_\mu A^\mu + 2A_\mu\partial^\mu\phi + \frac{M}{\sqrt{2}}\partial_\mu A^\mu = 0, \quad (t,x) \in \mathbf{R}^{1+n},$$

where $\Box = \partial_\mu\partial^\mu$. The constraint (1.6) is a gauge condition associated with the unitary gauge.

If $\phi > -M/\sqrt{2}$, we can rewrite (1.3) by using (1.6) as follows.

$$(1.7) \qquad \begin{aligned}(\Box + M^2)A^\mu &= 2\partial^\mu\big((\phi + M/\sqrt{2})^{-1}A_\nu\partial^\nu\phi\big) \\ &\quad - 2\sqrt{2}MA^\mu\phi - 2A^\mu\phi^2 \\ &= \frac{2\sqrt{2}}{M}\partial^\mu(A_\nu\partial^\nu\phi) - 2\sqrt{2}MA^\mu\phi \\ &\quad -2A^\mu\phi^2 + f(\phi, A_\nu, \partial^\nu\phi, \partial^\mu A_\nu, \partial^\mu\partial^\nu\phi), \quad (t,x) \in \mathbf{R}^{1+n},\end{aligned}$$