Approximations of the Relativistic Euler-Poisson-Darboux Equation

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1 Introduction and conclusions

In [2] we investigated the Cauchy problem to the relativistic Euler equation

$$(rE) \qquad \frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{(\rho+P/c^2)u}{1-u^2/c^2} = 0,$$

$$\frac{\partial}{\partial t}\frac{(\rho+P/c^2)u}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{P+\rho u^2}{1-u^2/c^2} = 0.$$

Here c is a positive constant, the speed of light, and P is a given smooth function of ρ satisfying the assumption

(A): $P(\rho) > 0, 0 < P' = dP/d\rho < c^2, 0 < P'' = d^2P/d\rho^2$ for $\rho > 0$, and

$$(1) \qquad P = A_0\rho^{5/3}(1 + \sum_{j=1}^{\infty} A_j(\rho^{2/3}/c^2)^j)$$

as $\rho \to 0$. Here A_0 is a positive constant and $\sum A_j z^j$ is a power series with positive radius of convergence.

In order to prove the existence of global weak solutions to the Cauchy problem by the theory of compensated compactness, we have to solve *the relativistic Euler-Poisson-Darboux equation*

$$(rEPD) \qquad \eta_{xx} - \eta_{yy} + A(x,y)\eta_y + B(x,y)\eta_x = 0,$$

where the independent variables are

$$(2) \qquad x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho$$

and the unknown function $\eta(x,y)$ is an entropy to (rE). The coefficients of $(rEPD)$ are given by

$$A(x,y) = \frac{1}{\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho+P/c^2}{2P'}P'')\frac{1+P'u^2/c^4}{1-P'u^2/c^4},$$

$$B(x,y) = -\frac{2u/c^2}{1-P'u^2/c^4}(1 - \frac{P'}{c^2} - \frac{\rho+P/c^2}{2P'}P'').$$

By the definition (2) and the assumption (1), we see that $A(x,y)$ and $B(x,y)$ are of the form

$$\begin{aligned} A(x,y) &= \frac{2}{y} + \epsilon y a(\epsilon x^2, \epsilon y^2) = \frac{2}{y} + \epsilon y \sum_{j+k\geq 0} a_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\ a_{00} &= -\frac{4}{9}(1 + \frac{7}{20}\frac{A_1}{A_0}), \\ B(x,y) &= -\frac{4}{3}\epsilon x b(\epsilon x^2, \epsilon y^2) = -\frac{4}{3}\epsilon x \sum_{j+k\geq 0} b_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\ b_{00} &= 1. \end{aligned}$$

Here and hereafter we denote

$$\epsilon = 1/c^2. \tag{3}$$

Introducing the new unknown V by

$$\frac{\partial \eta}{\partial y} = yV, \qquad \eta(x,y) = IV(x,y) = \int_0^y YV(x,Y)dY, \tag{4}$$

the singularity of $A(x,y)$ in $(rEPD)$ can be eliminated and $(rEPD)$ is reduced to the equation

$$V_{yy} - V_{xx} = \epsilon(yaV_y - \frac{4}{3}xbV_x + 2(a + \epsilon y^2 D_2 a)V - \frac{8}{3}\epsilon x D_2 b I V_x).$$

The problem

$$V_{yy} - V_{xx} = \epsilon(yaV_y - \frac{4}{3}xbV_x + 2(a + \epsilon y^2 D_2 a)V - \frac{8}{3}\epsilon x D_2 b I V_x), \tag{Q}$$

$$V|_{y=0} = 0, \qquad V_y|_{y=0} = 4\phi(x)$$

admits a unique solution V for any smooth ϕ given by a formula

$$V(x,y) = \int_{x-y}^{x+y} G(x,y,\xi - x, \epsilon)\phi(\xi)d\xi. \tag{5}$$

For the proof see [2], Section 5. $G(x,y,z,\epsilon)$ is a smooth function of $|x| < \infty, y \geq 0, |z| \leq y$. Therefore by defining

$$K(x,y,z,\epsilon) = JG(x,y,z,\epsilon) = \int_{|z|}^y YG(x,Y,z,\epsilon)dY, \tag{6}$$

we have a formula

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi - x,\epsilon)\phi(\xi)d\xi \tag{7}$$

for solutions of $(rEPD)$. We call this formula *the relativistic Darboux formula* and K *the relativistic Darboux kernel.* We know

$$K = (y^2 - z^2)(1 + O(\epsilon y)). \tag{8}$$

The purpose of this article is to study the properties of this relativistic Darboux kernel. The motivations are as follows.