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On uniqueness and existence of slowly decaying positive radial solutions for some semilinear elliptic equations

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1 Introduction

In this paper, we consider the following semilinear elliptic equation

$$\Delta u + f(|x|, u) = 0 \quad \text{in} \quad \mathbb{R}^n \ (n \geq 3). \tag{1}$$

Since we are only interested in positive radial solutions, we study the following problem :

$$u'' + \frac{n-1}{r}u' + f(r, u) = 0, \tag{2}$$

$$u > 0 \ (r > 0) \text{ and } u \to 0 \text{ as } r \to \infty, \tag{3}$$

where $r > 0$, $r = |x|$, $x \in \mathbb{R}^n$. If $f(r, u) > 0$ for small $u > 0$, then $r^{n-2}u(r)$ is increasing for large $r > 0$ (see the proof of Lemma 2.1), and therefore the solutions of (2)-(3) can be classified into two types:

(R) if $r^{n-2}u(r) \to C < \infty (r \to \infty)$ for some $C > 0$, then $u(r)$ is called a rapidly decaying solution,

(S) if $r^{n-2}u(r) \to \infty (r \to \infty)$, then $u(r)$ is called a slowly decaying solution.

Many authors studied the non-existence or existence and uniqueness of positive radial solutions of (2)-(3) under suitable structure conditions on $f(r, u)$ (see [Ni],[NS],[Pan1],[LN],[SZ],[Li],[Pan2] and the references therein). The case $f(r, u) = u^p$ has been extensively studied. Among them, it is known that if $n/(n-2) < p < (n+2)/(n-2)$, there exists only one slowly decaying positive solution u near infinity, namely $u(r) = \lambda r^{-\alpha}$ with $\alpha = 2/(p-1), \lambda = \{\alpha(n-2-\alpha)\}^{1/(p-1)}$ (see e.g. [SZ]). For the case $f(r, u) = u^p + u^q, n/(n-2) <$

$p < (n+2)/(n-2), p < q$, several authors studied existence and uniqueness of slowly decaying solutions of (2)-(3). In particular, Qi and Lu[QL] proved the existence and uniqueness of slowly decaying solution of (2)-(3) near infinity under the assumptions $n/(n-2) < p < (n+2)/(n-2), p < q$. Furthermore, they showed that if we impose the additional condition $q \leq (n+2)/(n-2)$, then the slowly decaying solution can be extended on $(0, \infty)$ as a singular solution, i.e. $\lim_{r \to 0} u(r) = +\infty$. See also [Pan1],[LN],[SZ] for some previous results on this problem. For the case $q > (n+2)/(n-2)$, the classification of the slowly decaying solution u as $r \to 0$ is difficult and has been an open problem. Very recently, partial results in this direction was obtained by R.Bamón, I.Flores and M.del Pino (see [BFP] for the details).
The purpose of this paper is to show uniqueness and existence of slowly decaying solutions near infinity for general nonlinearity $f(r,u)$ which satisfies certain structure conditions, including the typical one $f(r,u) = u^p + K(r)u^q$, but $K(r)$ is not necessarily bounded. Furthermore, we investigate the sufficient condition on $f(r,u)$ to make a slowly decaying solution singular at $r = 0$.
Throughout this paper, we assume that the constants p and q satisfy the following relations:

$$\frac{n}{n-2} < p < \frac{n+2}{n-2},\ p < q.$$

We also use the notation :

$$\alpha = \frac{2}{p-1},\ \sigma = (q-p)\alpha,\ \lambda = \{\alpha(n-2-\alpha)\}^{\frac{1}{p-1}}.$$

We assume the following conditions for $f(r,u)$.

(A-1) $f(r,u) = 0$ for $u \leq 0$ and $f(r,u)$ is continuous on $(0,\infty) \times (0,\infty)$ and locally Lipschitz continuous with respect to u.

(A-2) There exist positive constants C_0 and δ such that $f(r,u) \geq C_0 u^p$ holds for $u \in (0,\delta)$.

(A-3) There exists a function $K(r)$ such that $f(r,u) = u^p + f_1(r,u)$, $|f_1(r,u)| \leq K(r)u^q$ for $u \in (0,\delta)$ and $K(r) = O(r^l)$ $(r \to \infty)$ for some $l < \sigma$.

(A-4) There exists a function $\widetilde{K}(r)$ such that $|\{f_1(r,u)\}_u| \leq \widetilde{K}(r)u^{q-1}$ for $u \in (0,\delta)$ and $\widetilde{K}(r) = O(r^l)$ $(r \to \infty)$ for some $l < \sigma$.

First, we state the main result on the uniqueness of slowly decaying positive solutions near infinity.