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PERIODIC SOLUTIONS OF NONLINEAR EQUATIONS OF STRING WITH PERIODICALLY OSCILLATING BOUNDARIES

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Dedicated to Professor Kunihiko Kajitani on his sixties birthday

1. Introduction

In the study of the motions of nonlinear vibrating string with periodically oscillating ends, it seems to be interesting to investigate under which conditions periodic motions exist.

In this paper, we shall consider an oscillating string of finite length in the (x, u)-plane. Let the ends of the string move time-periodically on the (x, u)-plane and a nonlinear time-periodic vertical external force work on the string. We shall be concerned with *the existence of the time-periodic motions of the vibrating string under small vertical external forces.* This problem is mathematically formulated as the existence problem of periodic solutions of the Dirichlet boundary value problem for one-dimensional wave equation with a time-periodic nonlinear forcing term, where the boundaries oscillate periodically in t on the x-axis and the ends of the string are forced to move periodically in t in the vertical direction.

Let Ω be a time-periodic noncylindrical domain in (x, t)-plane defined by

$$a_1(t) < x < a_2(t), \quad t \in R^1.$$

Here $a_1(t)$ and $a_2(t)$ are periodic functions. The period is normalized to 1, for simplicity. Consider BVP (the boundary value problem) for a nonlinear one-dimensional wave equation :

$$\partial_t^2 u - \partial_x^2 u = \mu p(x, t) + f(x, t, u), \ \ (x, t) \in \Omega, \tag{1.1}$$

$$u(a_1(t), t) = \nu b_1(t), \quad u(a_2(t), t) = \nu b_2(t), \quad t \in R^1, \tag{1.2}$$

where $p(x, t)$ and $f(x, t, u)$, and $b_i(t)$, $i = 1, 2$, are periodic with period 1 in t, and $f(x, t, u)$ is of order more than or equal to 2 with respect to u. $p(x, t)$ and $f(x, t, u)$ satisfy some compatible boundary conditions (See (A4) later). As a typical example of f, if $b_i(t)$, $i = 1, 2$, identically vanish, then we give $f(x,t,u) = \pm u^m \, (m \geq 2)$. μ and ν are small parameters and are supposed to satisfy $\nu = \nu(\mu) = O(\mu) \, (\mu \to 0)$ continuous in μ. The above dependence of ν on μ is naturally imposed because we shall look for the small amplitude solutions and the external force working the whole string is of $O(\mu) \, (\mu \to 0)$. We assume that $a_1(t)$ and $a_2(t)$ satisfy $|\, a_i'(t) \,| < 1 \; (i = 1, 2)$. This condition is natural in the sense that the boundaries oscillate with slower speed than the eigenspeed 1 of waves by (1.1). Otherwise, the shock waves come out.

The aim of this paper is *to show the existence of time-periodic solutions with small amplitude of* BVP (1.1)-(1.2) *with the same period* 1 *as that of the given data.*

We define the following composed function A that is a fundamental tool in this research. Let A be a composed function defined by

$$A = A_1^{-1} \circ A_2, \quad A_i = (I + a_i) \circ (I - a_i)^{-1}, \quad i = 1, 2, \tag{1.3}$$

where I is an identity function, f^{-1} means the inverse function of f and $\circ$ means the composition operation of functions *i.e.* $f \circ g(x) = f(g(x))$. Geometrically A is a map naturally defined by the reflected characteristics in the (x,t)-plane. A is one dimensional periodic dynamical system. It is known in a series of works ([Ya1]-[Ya4], [Ya6]) that A and its rotation number $\rho(A)$ play an essential role in studying the qualitative behavior of solutions of IBVP and BVP in domain with periodically oscillating boundaries. For the definition of the rotation number, see *Notation and Definitions* in this section.

For the case where the ends of the string are fixed, BVP is of the form

$$\partial_t^2 u - \partial_x^2 u = F(x,t,u), \;\; (x,t) \in (0,a) \times R^1, \tag{1.4}$$

$$u(0,t) = u(a,t) = 0, \quad t \in R^1, \tag{1.5}$$

where a is a positive constant. In this case there are very many works on the existence of time-periodic solutions of BVP (1.4)-(1.5) (see [R1] [R2] [B-C-N] [W] etc. and see the references therein). It should be noted that the ratio of the period of the forcing term $F(x,t,u)$ to the length a of the interval $[\, 0, a \,]$ plays an important role in the study of the behavior of the solution. That is, the behaviors depend on the rationality or irrationality of the ratio. As is shown in [Ya8], even in the linear case *i.e.*, $F(x,t,u) = F(x,t)$ in (1.4) it happens that there are no bounded solutions, as a matter of course, no periodic solutions of (1.1)-(1.2) if the Diophantine order of the irrational ratio is large and