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Existence of global solutions to the Cauchy problem of Kirchhoff type quasilinear wave equation with weakly nonlinear dissipation

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1 Introduction

In this note we consider the existence of global solution to the initial value problem for the quasilinear wave equation:

$$u_{tt} - (1 + \|\nabla u\|^2)\Delta u + \rho(u_t) = 0 \quad \text{in} \quad R^N \times (0, \infty), \tag{1.1}$$

with

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x), \tag{1.2}$$

where $\|\cdot\|$ denotes L^2 norm in R^N and $\rho(v)$ is a differentiable function satisfying

$$0 < k_0 \le \rho'(v) \le k_1 < \infty \quad \text{and} \quad \rho(0) = 0. \tag{1.3}$$

For the Kirchhoff type quasilinear wave equation it is natural to seek for the solutions in the class $C([0,\infty); H_2) \cap C^1([0,\infty); H_1) \cap C^2([0,\infty); L^2)$ or a little weaker space $L^\infty([0,\infty); H_2) \cap W^{1,\infty}([0,\infty); H_1) \cap W^{2,\infty}([0,\infty); L^2)$ (cf. I. Lasiecka and J.Ong [2]), and we are interested in the global solution of the problem (1.1)-(1.2) in such a class, (we often call such a solution as H^2 solution). When $\rho(u_t) = u_t$, linear, we see $\rho(u_t)u = \frac{1}{2}\frac{d}{dt}|u|^2$ and by use of this fact we can easily derive the a priori estimates

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \le C(\|u_0\|_{H_1} + \|u_1\|_{L^2})(1 + t)^{-1}$$

and

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C(\|u_0\|_{H_2} + \|u_1\|)(1+t)^{-2}$$

if $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ is small. These a priori estimates are sufficient for the desired global solution. Indeed, K. Mochizuki [5] has proved such result under a more general condition on the dissipation. However, the proof heavily depends on the linearity of the dissipation $\rho(u_t)$ and cannot be applied to the case of nonlinear dissipation. Y. Yamada [9] proved the existence of global solutions without direct use of the decay properties. But in [9] also, the linearity of the dissipation is essentially used. The object of this note is to prove the existence of global H^2-solution when $\rho(v)$ is weakly nonlinear as (1.3), though in our case, solution $u(t)$ itself belongs to $L^\infty_{loc}([0,\infty); L^2)$, not $L^\infty([0,\infty); L^2)$.

Our proof is based on the following observations.

First, we see for an assumed H^2-solution $u(t)$,

$$E(t) + \int_0^t \int_{R^N} \rho(u_t(s))u_t(s)dxds = E(0),$$

where

$$E(t) = \frac{1}{2}[\|u_t(t)\|^2 + (1 + \|\nabla u(t)\|^2)\|\nabla u(t)\|^2]$$

and hence

$$k_0 \int_0^\infty \|u_t(s)\|^2 ds \le E(0) < \infty. \tag{1.4}$$

Next, differentiating the equation we have

$$u_{ttt} - (1 + \|\nabla u\|^2)\Delta u_t - 2(\nabla u, \nabla u_t)\Delta u + \rho'(u_t)u_{tt} = 0,$$

which is rewritten as

$$U_{tt} - (1 + \|\nabla u\|^2)\Delta U + \rho'(u_t)U_t = 2(\nabla u, \nabla u_t)\Delta u$$

with $U = u_t$. Since $\rho'(u_t)U_t$ is linear in U_t, we can expect the decay estimate

$$\|U_t(t)\|^2 + \|\nabla U(t)\|^2 \le C(\|u_0\|_{H_2} + \|u_1\|)(1+t)^{-1} \tag{1.5}$$

if (u_0, u_1) is small. This estimate is weaker than the case of linear dissipation, but combining this with (1.4) we have a hope to get desired H_2-solutions.

We notice that if the problem is considered in a bounded domain Ω with the boundary condition $u|_{\partial\Omega} = 0$, it is easy to derive exponential decay

$$E(t) \le C(E(0))e^{-\lambda t}, \quad \lambda > 0,$$

and the global existence of H_2- solution is easily proved. In fact, more general problem have been treated by many authors (Lasiecka and Ong [2], T.