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The L^2-boundedness of Pseudodifferential Operators with Simple Symbols

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1. Introduction

In this paper we improve a sufficient condition for the L^2-boundedness of pseudodifferential operators with simple symbols $p(x,\xi)$ of $S^0_{1,0}$-type. We want to weaken the smoothness assumption with respect to ξ as much as possible under some continuity in x.

Concerning the known results and the references in this direction, we refer to the paper [5] of Yamazaki, who treated the boundedness of product-type pseudodifferential operators with simple or double symbols in the weighted $L^p(\mathbb{R}^n)$-space with a multiple modulus of growth and continuity. In this paper we discuss only the boundedness of the (non-product-type) pseudodifferential operators $p(X,D)$ with simple symbols $p(x,\xi)$ in the (unweighted) $L^2(\mathbb{R}^n)$ with a modulus of continuity. In this case the best result was first obtained by Muramatu and Nagase [4]. It was generalized by Yamazaki [5] to the case of product-type operators and so on. Roughly speaking, the sufficient condition obtained in [4] or [5] means that $p(x,\xi)$ has the continuity in x expressed by a modulus of continuity $\omega(t)$ and that $p(x,\xi)$ belongs to the $C^{n/2+\varepsilon}$-class in ξ for any $\varepsilon>0$ and that the derivatives $\partial^\alpha_\xi p(x,\xi)$ satisfy some estimates.

The purpose of this paper is to show that the smoothness condition with respect to ξ can be relaxed. Our condition is expressed by a function $t^{n/2}\psi(t)$ defined on $(0,\infty)$, where $\psi(t)$ tends to 0 as $t\to 0$ more slowly than t^ε for any $\varepsilon>0$. In our terminology the smoothness of the $C^{n/2+\varepsilon}$-class function corresponds to $t^{n/2+\varepsilon}$. Hence the smoothness corresponding to $t^{n/2}\psi(t)$ is weaker than that of the $C^{n/2+\varepsilon}$-class function.

The method to prove our result is similar to that Yamazaki [5] employed. First we decompose the symbol $p(x,\xi)$ into the sum of the functions $p_k(x,\xi)$ with compact support in ξ-space and further decompose $p_k(x,\xi)$ into the

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sum of a regularized part $p_{k,0}(x,\xi)$ and the remainder part $p_{k,1}(x,\xi)$ so that the Fourier transform of the function associated with $p_{k,0}(X,D)$ has compact support. Then we estimate the kernels of the pseudodifferential operators $p_{k,a}(X,D)$ $(a=0,1)$, which is related to the inverse Fourier transform of $p_{k,a}(x,\xi)$ with respect to ξ. Finally we apply the lemmas concerning the Littlewood-Paley decomposition.

There are two keys in the proof of our result. One key is the Sobolev space with function parameter due to Muarmatu [3], which enables us to express the smoothness condition on the symbol with respect to ξ by a function $t^{n/2}\psi(t)$. Another key is no use of the lemma concerning the strong maximal function which Yamazaki used in order to treat the L^p case. Instead of this lemma we use Fubini's theorem which is applicable only in the L^2 case.

The outline of this paper is as follows. In Sectin 2 we state the main theorem after introducing the weight function. In Section 3 we restate the smoothness condition in terms of difference operators. In Section 4 we decompose the symbol and estimate the integral kernel of the pseudodifferential operator with help of the Sobolev space with function parameter. Finally, in Section 5, we complete the proof of the main theorem.

2. Statement of the main theorem

Let $\mathbb{N}$ and $\mathbb{R}_+$ denote the set of non-negative integers and that of non-negative real numbers respectively. We often abbrebiate $L^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}$ to L^2 and $\int$ respectively. The Fourier transform is defined by $\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int \exp(-ix\xi)u(x)\,dx$, and the inverse Fourier transform is defined by $\mathcal{F}^{-1}u(x) = c_n \int \exp(ix\xi)u(\xi)\,d\xi$, where $c_n = (2\pi)^{-n}$. We set $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

The pseudodifferential operator with simple symbol $p(x,\xi)$ is given by

$$p(X,D)u(x) = c_n \int e^{ix\xi} p(x,\xi)\widehat{u}(\xi)\,d\xi = c_n \iint e^{i(x-y)\xi} p(x,\xi)u(y)\,dy\,d\xi,$$

where the last integral is interpreted as an oscillatory integral. We define difference operators Δ_z and $\Delta_{2,\zeta}$ by

$$\Delta_z p(x,\xi) = p(x+z,\xi) - p(x,\xi),$$
$$\Delta_{2,\zeta} p(x,\xi) = p(x,\xi+\zeta) - p(x,\xi).$$

We regard Δ_z^0 and $\Delta_{2,\zeta}^0$ as the identity operators.

In our theorem the condition on the smoothness of $p(x,\xi)$ with respect to ξ is expressed by the weight function, which we introduce here briefly by