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Decomposition of Variation of Constants Formula for Abstract Functional Differential Equations

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1. Introduction

In this paper we are concerned with the linear functional differential equation

$$\dot{u}(t) = Au(t) + F(t)u_t + f(t) \tag{1}$$

on a phase space $\mathcal{B} = \mathcal{B}((-\infty, 0]; \mathbb{X})$ satisfying some fundamental axioms listed in Section 2.1, where A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Banach space $\mathbb{X}$, u_t is an element of $\mathcal{B}$ defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$, $F(t)$ is a bounded linear operator mapping $\mathcal{B}$ into $\mathbb{X}$ which depends strongly continuously and periodically on t, and f is an $\mathbb{X}$-valued bounded and continuous function.

In a recent paper [24], Murakami, Naito and Nguyen have established a variation of constants formula (VCF) in the phase space for Eq. (1). The formula has been then applied to extend a classical theorem of Massera [22] on the existence of periodic solutions of linear ordinary differential equations to almost periodic solutions for Eq. (1).

A key point in [24] is to analyze difference equations associated with Eq. (1), which are derived naturally from the formula.

In this paper we will continue to study the subject, and establish several sharper results on the existence of almost periodic and quasiperiodic solutions for Eq. (1). Our approach employed in this paper is different from the one in [24]. Indeed, we will decompose the variation of constants formula into two parts referred to as the *stable part* of VCF and the *unstable part* of VCF (Theorem 3.1), and study each part of VCF to ensure the existence of almost periodic solutions and quasiperiodic solutions for Eq. (1). There are several advantages in our approach. Among them, we point out the following two facts: Roughly speaking, some spectral properties of the function f is inherited to the stable part of VCF (Theorem 3.3). Meanwhile, the unstable part of VCF is reduced to an ordinary

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differential equations (Theorem 3.5), and via this fact some spectral properties of f is inherited also to the unstable part of VCF (Theorem 3.6).

As an appendix, we refer to the Riesz representation for each element belonging to the dual space of $\mathcal{B}((-\infty]; \mathbb{X})$ and Favard's type argument to ensure the existence of almost periodic solutions of ordinary differential equations with a discontinuous forcing term. They seem to be unknown in the general situation and will be indispensable in our approach.

2. Abstract Functional Differential Equations

Throughout this paper, we will use the following notation. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, real numbers and complex numbers, respectively. Also, $C(J, \mathbb{X})$ denotes the space of all $\mathbb{X}$-valued continuous functions on J, and $BC(J, \mathbb{X})$ denotes the subspace of $C(J, \mathbb{X})$ consisting of all bounded and continuous functions on J.

We now consider the abstract functional differential equation

$$\frac{du(t)}{dt} = Au(t) + F(t)u_t + f(t), \tag{1}$$

where A is the generator of a semigroup of linear operators on a Banach space $\mathbb{X}$, $F(t)$ is a bounded linear operator from $\mathcal{B}$ into $\mathbb{X}$ which is periodic in t with period 1, where $\mathcal{B}$ is a fading memory phase space of Eq. (1) with infinite delay satisfying the axioms listed below and $f \in BC(\mathbb{R}, \mathbb{X})$. We emphasize that the assumption that the period of F is 1 does not constitute any restrictions on the obtained results.

2.1. Fading Memory Phase Spaces.

We will give a precise definition of the notion of fading memory space for Eq. (1) in this subsection. Let us denote the norm of $\mathbb{X}$ by $\|\cdot\|_{\mathbb{X}}$. For any function $x : (-\infty, a) \mapsto \mathbb{X}$ and $t < a$, we define a function $x_t : \mathbb{R}^- := (-\infty, 0] \mapsto \mathbb{X}$ by $x_t(s) = x(t+s)$ for $s \in \mathbb{R}^-$. A Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ which consists of functions $\psi : (-\infty, 0] \mapsto \mathbb{X}$ is called a *fading memory space* if it satisfies the following axioms:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $\mathbb{R}^+$ with the property that if $x : (-\infty, a) \mapsto \mathbb{X}$ is continuous on $[\sigma, a)$ with $x_\sigma \in \mathcal{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

- (i) $x_t \in \mathcal{B}$,
- (ii) x_t is continuous in t (w.r.t. $\|\cdot\|_{\mathcal{B}}$),
- (iii) $N\|x(t)\|_{\mathbb{X}} \le \|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}\|x(s)\|_{\mathbb{X}} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

(A2) If $\{\phi^k\}$, $\phi^k \in \mathcal{B}$, converges to ϕ uniformly on any compact set in $\mathbb{R}^-$ and if $\{\phi^k\}$ is a Cauchy sequence in $\mathcal{B}$, then $\phi \in \mathcal{B}$ and $\phi^k \to \phi$ in $\mathcal{B}$.

A fading memory space $\mathcal{B}$ is called a *uniform fading memory space*, if it satisfies (A1) and (A2) with $K(\cdot) \equiv K$ (a constant) and $M(\beta) \to 0$ as $\beta \to \infty$ in (A1). A typical example