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On White Hole Solutions of a Class of Nonlinear Ordinary Differential Equations of the Second Order

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1. Introduction

Consider the nonlinear ordinary differential equation

(A) $$(|y'|^{\alpha})' + q(t)|y|^{\beta} = 0$$

where $\alpha, \beta \in R, \alpha \neq 0$, are constants and $q : [a, \infty) \to (0, \infty), a \geq 0$, is a continuous function.

By a solution of (A) on an interval $J = [t_0, T), a \leq t_0 < T \leq \infty$, we mean a function $y \in C^1(J)$ which has the property $|y'|^{\alpha} \in C^1(J)$ and satisfies (A) at each point of J. If we denote by T_y the maximal existence time of y, then we say that $y(t)$ is *proper* if $T_y = \infty$ and

$$\sup\{|y(t)| : t \in [\tau, \infty)\} > 0 \quad \text{for} \quad \text{any} \quad \tau \geq t_0.$$

A solution $y(t)$ is called *singular* if either $T_y < \infty$ or $T_y = \infty$ and there exists $T \in [t_0, \infty)$ such that

$$\max\{y(s) : t \leq s \leq T\} > 0 \quad \text{for} \quad t \in (t_0, T)$$

and $y(t) = 0$ for $t \geq T$. In the later case, the interval $[t_0, T)$ is called the *support* of the solution $y(t)$.

Our main objective here is to investigate the structure of the solution set of (A) in the case $\alpha > 0$ and to show that nonlinear equations of the form (A) may have singular solutions of a new type satisfying

(1) $$\lim_{t \to T_y - 0} y(t) = \text{const} \neq 0 \quad \text{and} \quad \lim_{t \to T_y - 0} y'(t) = 0$$

at the (finite) right end-point of the maximal interval of existence. By analogy with the concept of "black hole" solutions, that is, singular solutions defined on $[t_0, T_y)$ and satisfying

$$(2) \qquad \lim_{t\to T_y-0} y(t) = \text{const} \neq 0 \quad \text{and} \quad \lim_{t\to T_y-0} |y'(t)| = \infty,$$

introduced by the present authors in [3] (see also [4], [7] and [8]), positive solutions of (A) satisfying (1) as t approaches the maximal existence time $T_y < \infty$ are called *white hole* (singular) *solutions.*

An example of a nonlinear equation of the form (A) (with $\beta = 0$) which possesses singular solutions of this new type is

$$(3) \qquad (|y'|^\alpha)' + \left(\frac{\alpha+1}{\alpha}\right)^\alpha = 0,$$

where $\alpha > 0$. Indeed, for any given $T > a$ and $c > 0$, the function $y(t) = c + (T-t)^{\frac{\alpha+1}{\alpha}}$ defined and positive on $[a, T)$ is a decreasing solution of (3) with a singularity of white hole type at T.

Similarly, for any $c > 0$, the function $y(t) = c - (T-t)^{\frac{\alpha+1}{\alpha}}$ provides an example of a 'local' increasing white hole solution of (3) which is defined and positive in some sufficiently small left neighborhood of the maximal existence time T.

Another simple example of an equation of the form (A) having white hole singular solutions is the following "almost linear" equation

$$(4) \qquad (|y'|)' + |y| = 0.$$

As easily seen, for any real T and any $c > 0$, the function $y(t) = c\cos(T-t)$ defined and positive on $[t_0, T), t_0 \geq T - \frac{\pi}{2}$, is an increasing singular solution of (4) which is of the white hole type.

While the existence and asymptotic theory for quasilinear second-order differential equations of the form

$$(B) \qquad (p(t)|y'|^\alpha \text{sgn} y')' + q(t)|y|^\beta \text{sgn} y = 0, \quad t \geq a,$$