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Blowup rate of solutions to the Brezis-Nirenberg equations with the Robin condition

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1 Introduction

The aim of this paper is to investigate the blowup behavior of solutions to the Brezis-Nirenberg equation with the Robin condition. In our previous paper Kabeya, Yanagida and Yotsutani [10], we proved the range of λ for which a unique positive radial solution to

$$\begin{cases} \Delta u + \lambda u + u^5 = 0 \quad \text{in} \quad B = \{x \in \mathbf{R}^3 : \ |x| < 1\}, \\ u > 0 \quad \text{in } B, \\ \kappa \dfrac{\partial u}{\partial \nu} + u = 0 \quad \text{on } \partial B, \end{cases} \tag{1.1}$$

exists, where ν is the outward unit normal vector to ∂B, $\lambda < \lambda_0$ (λ_0 is the first eigenvalue of $-\Delta$ with the homogeneous Robin condition on B, $\lambda_0 = \pi^2$ if $n = 3$ and $\kappa = 0$) for each $\kappa \geq 0$.

When $\kappa = 0$, in the three dimensional case, a solution to (1.1) exists for $\pi^2/4 < \lambda < \pi^2$ while in the higher dimension, a solution does for $0 < \lambda < \lambda_0$ (see e.g., Brezis and Nirenberg [4], Brezis and Peletier [5], or [10]). In this sense, the three dimensional case is an exceptional case and interesting

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phenomena occur in this case. So we concentrate on the three dimensional case.

Since our concern is on radial solutions, we consider the initial value problem of the ordinary differential equation

$$\begin{cases} u_{rr} + \dfrac{2}{r} u_r + \lambda u + u_+^5 = 0, \quad 0 < r < 1, \\ u(0) = \alpha, \ u_r(0) = 0, \end{cases} \tag{1.2}$$

and seek a suitable number $\alpha > 0$ satisfying $u(r) > 0$ on $(0,1)$ and

$$\kappa u_r(1) + u(1) = 0, \tag{1.3}$$

where $u_+ = \max\{u, 0\}$. Note that (1.2) has a solution for any $\alpha > 0$ and λ.

We introduce three numbers. Let $\mu_0 = \mu_0(\kappa) \in (0, \pi]$ be defined by

$$\begin{cases} \mu_0 = \pi, & \text{if} \quad \kappa = 0, \\ 1 - \mu_0 \cot \mu_0 = \dfrac{1}{\kappa}, & \text{if} \quad \kappa > 0. \end{cases}$$

Note that μ_0^2 is the first radial eigenvalue of $-\Delta$ subject to the boundary condition $\kappa \partial u / \partial \nu + u = 0$. For $0 \le \kappa \le 1$, define $\mu_1 = \mu_1(\kappa) \in [0, \pi/2]$ and $\zeta = \zeta(\kappa) \in [0, \infty)$ by

$$\begin{cases} \mu_1 = \dfrac{\pi}{2}, & \text{if} \quad \kappa = 0, \\ 1 + \mu_1 \tan \mu_1 = \dfrac{1}{\kappa}, & \text{if} \quad 0 < \kappa \le 1, \end{cases}$$

and

$$\begin{cases} \zeta = \infty, & \text{if} \quad \kappa = 0, \\ \zeta \coth \zeta = \dfrac{1}{\kappa}, & \text{if} \quad 0 < \kappa < 1, \\ \zeta = 0, & \text{if} \quad \kappa = 1, \end{cases}$$

respectively. As we will see in Theorem B, $\lambda = \mu_1^2$ is a blowup point. Also note that

$$\begin{cases} \Delta u + \mu_1^2 u = 0 \text{ in } B, \\ \kappa u'(1) + u(1) = 0 \end{cases}$$

has a positive singular solution.