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Notes on Space-Time Decay Properties of Nonstationary Incompressible Navier-Stokes Flows in $\mathbb{R}^n$

By

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Dedicated to Professor Shinnosuke Oharu on his 60th birthday

1. Introduction and results

Consider the nonstationary Navier-Stokes system in $\mathbb{R}^n$, $n \geq 2$, in the form of the integral equation:

$$\text{(IE)} \qquad u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)ds.$$

Here $\{e^{-tA}\}_{t\geq 0}$ is the heat semigroup, P is the bounded projetion from the space $\boldsymbol{L}^q$, $1 < q < \infty$, of vector fields to the subspace of $\boldsymbol{L}^q$ consisting of all divergence-free vector fields. As is shown in [4], the operator $e^{-tA}P\nabla\cdot$ has the kernel function $F = (F_{\ell,jk})^n_{j,k,\ell=1}$ with

$$F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j\partial_k\partial_\ell E_{s+t}(x)ds, \qquad \partial_j = \partial/\partial x_j,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $t > 0$, and $E_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}$.

Inspired by Takahashi [9], we proved in [4] the following

Theorem 1. *Fix $1 \leq \gamma \leq n+1$ and let a be divergence-free, satisfying*

$$(1.1) \qquad |(e^{-tA}a)(x)| \leq c(1+|x|)^{-\gamma}, \qquad |(e^{-tA}a)(x)| \leq c(1+t)^{-\frac{\gamma}{2}}.$$

If $c > 0$ is small, then (IE) *possesses a solution u such that, with another $c > 0$,*

$$|u(x,t)| \leq c(1+|x|)^{-\gamma}, \qquad |u(x,t)| \leq c(1+t)^{-\frac{\gamma}{2}}.$$

However, in proving Theorem 1, the Hardy space theory was used to find relevant decay estimates in t. In this paper we show that Theorem 1 can be deduced by a more elementary method, i.e., without using the Hardy space theory.

In [4] we also gave a class of initial data a for which $e^{-tA}a$ satisfies (1.1). But, the assumption on a imposed in [4] was too complicated and restrictive. So, we here give a simpler version of the assumption on a that ensures the validity of (1.1). More specifically, we shall show the following

Theorem 2. *Let a be divergence-free and satisfy*

$$c_0 = \sup_y (1+|y|)^\gamma |a(y)| < \infty \qquad \textit{for some } 0 < \gamma \leq n+1.$$

Suppose further that

$$c_1 = \int |a(y)|dy < \infty \quad \textit{when } \gamma = n, \qquad c_1 = \int |y||a(y)|dy < \infty \quad \textit{when } \gamma = n+1.$$

Then

$$|(e^{-tA}a)(x)| \leq cd(1+|x|)^{-\gamma}, \qquad |(e^{-tA}a)(x)| \leq cd(1+t)^{-\frac{\gamma}{2}}. \tag{1.2}$$

Here, $d = c_0 + c_1$ when $\gamma = n$ or $\gamma = n+1$; and $d = c_0$ otherwise.

Starting from Theorems 1 and 2, we deduced in [3] a space-time asymptotic expansion of solutions u in the case $\gamma = n+1$, which was then applied in [8] to find a lower bound estimate of rates of energy decay for weak solutions of (IE).

Finally, we supplement the recent result of Brandolese [2] on the existence of solutions which decay more rapidly than those treated in Theorem 1. Consider the divergence-free vector fields a such that

(a) a_j is odd in x_j and is even in each of the other variables.

(b) a is cyclically symmetric in the sense that

$$a_1(x_1, \ldots, x_n) = a_2(x_n, x_1, \ldots, x_{n-1}) = \cdots = a_n(x_2, \ldots, x_n, x_1).$$

Brandolese [2] shows the existence of a solution u satisfying (a) and (b) above, with the estimates

$$|u(x,t)| \leq c(1+|x|)^{-n-2}, \qquad |u(x,t)| \leq c(1+t)^{-\frac{n+2}{2}}. \tag{1.3}$$

Observe that the solutions above decay more rapidly than those treated in Theorem 1. But, estimate (1.3) seems not optimal. Indeed, we shall prove

Theorem 3. *Suppose a satisfies (a) and (b), and*

$$c_0 = \sup(1+|y|)^{n+3}|a(y)| < \infty, \qquad c_1 = \int |y|^3 |a(y)|dy < \infty. \tag{1.4}$$

(i) *The function $x \mapsto e^{-tA}a(x)$ satisfies (a) and (b), and there hold the estimates*

$$|(e^{-tA}a)(x)| \leq c(c_0+c_1)(1+|x|)^{-n-3}, \qquad |(e^{-tA}a)(x)| \leq c(c_0+c_1)(1+t)^{-\frac{n+3}{2}}.$$