Funkcialaj Ekvacioj
Volume 45, No. 2, pp 259--270

Small Data Global Existence of Solutions for Dissipative Wave Equations in an Exterior Domain

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Abstract

First we shall derive the basic decay estimates of the total energy and L^2-norm of a solution to the mixed problem for the linear dissipative wave equation in an exterior domain with the initial data satisfying some further restrictions as $|x| \to +\infty$. That decay estimates are faster than the usual one. Second we shall apply the decay estimates above to the exterior mixed problem of the semilinear dissipative wave equation in an exterior domain and we shall derive the small data global existence property to that problem with the power satisfying $1 + 4/(N+2) < p \leq N/(N-2)(N = 3, 4, 5)$ on the nonlinear term $|u|^p$.

1 Introduction

Let $\Omega \subset R^N (N \geq 3)$ be an exterior domain with compact smooth boundary $\partial\Omega$. Without loss of generality we may assume $0 \notin \bar{\Omega}$. In this paper we are concerned with the initial-boundary value problem for the linear dissipative wave equation:

$$v_{tt}(t,x) - \Delta v(t,x) + v_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega, \tag{1.2}$$

$$v|_{\partial\Omega} = 0, \quad t \in (0,\infty), \tag{1.3}$$

and the semilinear dissipative wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = |u(t,x)|^p, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.4}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.5}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.6}$$

Throughout this paper, $\|\cdot\|_q$ and $\|\cdot\|_{H^1}$ mean the usual $L^q(\Omega)$-norm and $H_0^1(\Omega)$-norm, respectively, and in particular, we set $\|\cdot\| = \|\cdot\|_2$ for simplicity. Furthermore, we adopt

$$(f,g) = \int_\Omega f(x)g(x)dx$$

as the usual $L^2(\Omega)$-inner product. The total energy $E_v(t)$ to the equation (1.1) and (1.4) is defined by

$$E_v(t) = \frac{1}{2}\|v_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla v(t,\cdot)\|^2.$$

The first purpose of this job is to derive certain decay estimates for the total energy $E_v(t)$ and the L^2-norm of a solution $v(t,x)$ to the linear problem (1.1)-(1.3) faster than the usual one through the (modified) time integral method developed in Ikehata-Matsuyama [3]. In that occasion, we do assume some further restrictions on the initial data as $|x| \to +\infty$. On the contrary, Ikehata-Matsuyama [3] and Saeki-Ikehata [11] adopted another weight condition on the initial data. For the exterior mixed problem, these restrictions on the initial data seem to be new (for conditions on the initial data with the compact support, see Dan-Shibata [1]). For these restrictions as $|x| \to +\infty$ on the initial data to the "Cauchy problem" of the equation (1.1), there are lots of related results and we refer the reader to Kawashima-Nakao-Ono [6], Matsumura [8] and the references thererein.

The second purpose of this paper is to determine the exponent p of the semilinear exterior problem (1.4)-(1.6) for which the small data global existence property holds. Very recently, in Ikehata-Miyaoka-Nakatake [4] and Todorova-Yordanov [12] they have derived such a critical (Fujita type) exponent $p_c = 1 + 2/N$ to the Cauchy problem of (1.4) in the framework of $L^1 \times L^1$ assumption on the initial data and of the initial data with compact support, respectively (for another type of critical exponents like $p_c = 1 + 2m/N$ for the Cauchy problem of (1.4) with $L^m \times L^m$ assumption on the initial data, see also Ikehata-Ohta [5]). These works are fully based on the decay estimates for the linear equations due to Matsumura [8] and Kawashima-Nakao-Ono [6]. Thus, it seems to be difficult to apply those decay estimates for the linear equations due to [6] and [8] to the present exterior mixed problem (1.4)-(1.6). On the other hand, in the framework of the compactly supported initial data Ikehata [2] has already constructed a small global solution to the exterior problem (1.4)-(1.6) with the power $1 + 6/(N+2) < p \leq N/(N-2)(N=3)$ or $1 + 6/(N+2) < p < +\infty(N=2)$. His result is based on the decay estimates for the linear equations which are developed in [3] and [11]. By using decay estimates for the linear equations developed in the former part instead of those developed in [3] and [11], we can exclude the compactness of the support on the initial data as in [2] to the problem (1.4)-(1.6) with further relaxed exponent, and we can also treat the higher dimensional case $N = 4, 5$ (for another exponent $p_c = 1 + 4/N$, see Nakao-Ono [10]).

In the following, we set

$$I_{0,u} = \|u_0\|_{H^1} + \|u_1\| + \|u_0 + u_1\|_{2N/(N+2)}.$$

Then our first result reads as follows.

Theorem 1.1 *Let $N \geq 3$. For each $[v_0, v_1] \in (H_0^1(\Omega) \cap L^{2N/(N+2)}(\Omega)) \times (L^2(\Omega) \cap L^{2N/(N+2)}(\Omega))$, the weak solution $v \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ to the linear problem (1.1)-(1.3) satisfies the decay estimates:*

$$\|v(t,\cdot)\|^2 \leq C I_{0,v}^2 (1+t)^{-1},$$

$$\|v_t(t,\cdot)\|^2 + \|\nabla v(t,\cdot)\|^2 \leq C I_{0,v}^2 (1+t)^{-2} \tag{1.7}$$

with some generous constant $C > 0$.