Funkcialaj Ekvacioj
Volume 45, No. 2, pp 237--258

Bäcklund Transformations and the Manifolds of Painlevé Systems

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1 Introduction

The purpose of this paper is to show that we can take coordinate systems determined by the Bäcklund transformations as coordinate systems of the manifolds of Painlevé systems constructed by K. Okamoto ([10]) (except the first one) and that the manifolds with parameters equivalent under the corresponding affine Weyl groups are mutually isomorphic.

The J-th Painlevé system ($J = II, III, IV, V, VI$) which is equivalent to the J-th Painlevé equation is the following Hamiltonian system

$$(H_{J,\alpha}) \qquad \delta q = \{H_J(q,p,t,\alpha), q\}, \quad \delta p = \{H_J(q,p,t,\alpha), p\},$$

where $\delta = d/dt$ for $J = II, IV$, $\delta = td/dt$ for $J = III, V$, $\delta = t(t-1)d/dt$ for $J = VI$, $\{\cdot,\cdot\}$ is the Poisson bracket defined by

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p},$$

and the Hamiltonian $H_J(q,p,t,\alpha)$, $\alpha = (\alpha_0, \alpha_1, ...)$ being parameters with a relation, is given by

$$\begin{aligned}
H_{II}(q,p,t,\alpha) &= \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - \alpha_1 q \\
&\quad (\alpha_0 + \alpha_1 = 1), \\
H_{III}(q,p,t,\alpha) &= q^2 p(p-1) + q[(\alpha_0+\alpha_2)p - \alpha_0] + tp \\
&\quad (\alpha_0 + 2\alpha_1 + \alpha_2 = 1), \\
H_{IV}(q,p,t,\alpha) &= qp(p - q - 2t) - 2\alpha_1 p - 2\alpha_2 q \\
&\quad (\alpha_0 + \alpha_1 + \alpha_2 = 1), \\
H_V(q,p,t,\alpha) &= q(q-1)p(p+t) - (\alpha_1+\alpha_3)qp + \alpha_1 p + \alpha_2 tq \\
&\quad (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1),
\end{aligned}$$

$$\begin{aligned}H_{VI}(q,p,t,\alpha) &= q(q-1)(q-t)p^2 - [(\alpha_0-1)q(q-1) + \alpha_4(q-1)(q-t) \\ &\quad +\alpha_3 q(q-t)]p + \alpha_2(\alpha_1+\alpha_2)(q-t) \\ &\quad (\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1).\end{aligned}$$

We notice that the forms of the Hamiltonians for $J = III, IV, V$ given here are slightly different from those in [2],[14],[4]. The Hamiltonian for $J = III$ or IV or V is obtained from that for $J = III'$ or IV or V in [2] respectively by certain change of variables (see Section 3).

Each Painlevé system determines a complex one dimensional nonsingular foliation of $\mathbf{C}^2 \times B_J (\ni (q,p,t))$ where

$$B_{II} = B_{IV} = \mathbf{C}, \quad B_{III} = B_V = \mathbf{C} - \{0\}, \quad B_{VI} = \mathbf{C} - \{0,1\}.$$

The system is holomorphically extended to one on a manifold $E_{J,\alpha}$ which is a fiber space over B_J having the $\mathbf{C}^2 \times B_J$ as a fiber subspace and the extended system defines a *uniform* foliation $\mathcal{F}_{J,\alpha}$ of $E_{J,\alpha}$ although the foliation of the $\mathbf{C}^2 \times B_J$ is not uniform ([14],[4],[10]). Here the *uniformity* of the foliation $\mathcal{F}_{J,\alpha}$ means that, *for any point $P_0 \in E_{J,\alpha}$, every curve in B_J starting from $\pi_J(P_0)$ is lifted on the leaf passing through P_0, where π_J is the projection from $E_{J,\alpha}$ to B_J.* We notice that the *uniformity* of the foliation is equivalent to the so-called *Painlevé property* for the Painlevé system, that is, *if $(q(t), p(t))$ is a local solution of $(H_{J,\alpha})$ determined by an arbitrary initial condition $q(t_0) = q_0 \in \mathbf{C}$, $p(t_0) = p_0 \in \mathbf{C}$ with $t_0 \in B_J$, then both $q(t)$ and $p(t)$ can be meromorphically continued along any curve in B_J with a starting point t_0.* The fibers of $E_{J,\alpha}$ are called the *spaces of initial conditions*([10]). Each $E_{J,\alpha}$ is described by the original chart $\mathbf{C}^2 \times B_J$ and a finite number of copies $\mathbf{C}_i^2 \times B_J$ of $\mathbf{C}^2 \times B_J$ where coordinate transformations are certain birational symplectic ones ([14],[4]).

On the other hand, each Painlevé system admits a Bäcklund transformation group of certain birational symplectic transformations each of which preserves the form of the Hamiltonian and changes the parameters α_i as an element of an affine Weyl group ([6],[7],[8],[9]). This fact was first recognized by K. Okamoto ([11]), but our presentation in the following is different from his.

Let $K = \mathbf{C}(q,p,t,\alpha)$ $(\alpha = (\alpha_0, \alpha_1, \ldots)$) be a differential field of rational functions of q, p, t, α with a derivation δ defined by

$$\delta f = \frac{\partial f}{\partial q} \cdot \{H_J(q,p,t,\alpha), q\} + \frac{\partial f}{\partial p} \cdot \{H_J(q,p,t,\alpha), p\} + \delta' f, \quad f \in K,$$

where δ' is $\partial/\partial t$ for $J = II, IV$, $t\partial/\partial t$ for $J = III, V$, and $t(t-1)\partial/\partial t$ for $J = VI$. (Notice that $\delta\alpha_i = 0$.) Then, there is a Bäcklund transformation group W which is a lift of an affine Weyl group acting on the α-space such that

(i) each $w \in W$ is an isomorphism from the field K to itself,
(ii) $\delta w = w\delta$, for $w \in W$,
(iii) $w\{f,g\} = \{w(f), w(g)\}$ for $w \in W$, $f, g \in K$.