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EXISTENCE OF POSITIVE SOLUTIONS TO SEMIPOSITONE FREDHOLM INTEGRAL EQUATIONS

Ravi P. Agarwal[1], Said R. Grace[2] and Donal O'Regan[3]

[2]Department of Engineering Mathematics, Cairo University, Orman, Giza 12221, Egypt
[3]Department of Mathematics, National University of Ireland, Galway, Ireland.

Abstract. New existence theorems are presented for semipositone integral equations of the form $y(t) = \mu \int_0^1 k(t,s)\, f(s, y(s))\, ds$ for $t \in [0,1]$. An application to second order boundary value problems is also discussed.

1. Introduction.

This paper presents three new existence results for semipositone Fredholm integral equations of the form

$$y(t) = \mu \int_0^1 k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0,1], \tag{1.1}$$

where $\mu > 0$ is a constant. Existence in both $C[0,1]$ and $L^p[0,1]$ will be discussed. Throughout this paper k is nonnegative but our nonlinearity f may take negative values. Problems of this type are referred to as semipositone problems in the literature and they arise naturally in chemical reactor theory [4]. The constant μ is called the Thiele modulus and of physical interest is the existence of positive solutions to (1.1) when $\mu > 0$ is small. The literature on positive solutions to Fredholm integral equations (see [3–8] and the references therein) is almost totally devoted to (1.1) when f takes nonnegative values (i.e. positone problems). Only a few results (see [1 Chapter 4]) are available for the semipositone problem.

Existence in this paper will be established using Krasnoselskii's fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 1.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$ and let $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$ be continuous and completely continuous. In addition suppose either*

$$\|A\,u\| \le \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_1 \ \ \textit{and} \ \ \|A\,u\| \ge \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_2$$

or

$$\|A\,u\| \ge \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_1 \ \ \textit{and} \ \ \|A\,u\| \le \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_2$$

hold. Then A has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

2. Semipositone problems.

In this section we present three new results for the semipositone Fredholm integral equation

$$y(t) = \mu \int_0^1 k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0,1]; \tag{2.1}$$

here $\mu > 0$ is a constant. Of physical interest is the existence of nonnegative solutions which are positive a.e. on $[0,1]$.

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$\begin{cases} \text{there exists } a \in C[0,1] \text{ and } t^\star \in [0,1] \text{ with } a(t) > 0 \\ \text{for a.e. } t \in [0,1] \text{ and } a(t^\star) > 0, \text{ there exists } \kappa \in L^1[0,1] \\ \text{with } \kappa(t) \geq 0 \text{ a.e. } t \in [0,1] \text{ and } \int_0^1 \kappa(s)\, ds > 0 \text{ such} \\ \text{that } a(t)\, \kappa(s) \leq k(t,s) \text{ for all } t \in [0,1], \text{ a.e. } s \in [0,1] \end{cases} \tag{2.2}$$

$$k_t(s) = k(t,s) \leq \kappa(s) \quad \text{for all } t \in [0,1], \quad \text{a.e. } s \in [0,1] \tag{2.3}$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0,1] \text{ to } L^1[0,1] \tag{2.4}$$

$$\begin{cases} f : [0,1] \times [0,\infty) \to \mathbf{R} \text{ is continuous and there} \\ \text{exists a constant } M > 0 \text{ with } f(t,u) + M \geq 0 \\ \text{for } (t,u) \in [0,1] \times [0,\infty) \end{cases} \tag{2.5}$$

$$\begin{cases} f(t,u) + M \leq \psi(u) \text{ on } [0,1] \times [0,\infty) \text{ with} \\ \psi : [0,\infty) \to [0,\infty) \text{ continuous and nondecreasing} \\ \text{and } \psi(u) > 0 \text{ for } u > 0 \end{cases} \tag{2.6}$$

$$\exists\, C > 0 \text{ with } \int_0^1 k(t,s)\, ds \leq C\, a(t) \text{ for } t \in [0,1] \tag{2.7}$$

$$\exists\, r \geq \mu\, M\, C \text{ with } \frac{r}{\psi(r)} \geq \mu \sup_{t \in [0,1]} \int_0^1 k(t,s)\, ds \tag{2.8}$$

$$\begin{cases} f(t,u) + M \geq g(u) \text{ for } (t,u) \in [0,1] \times (0,\infty) \text{ with} \\ g : [0,\infty) \to [0,\infty) \text{ continuous and nondecreasing} \\ \text{and } g(u) > 0 \text{ for } u > 0 \end{cases} \tag{2.9}$$

and

$$\exists\, R > r \text{ with } R \leq \mu \int_0^1 k(t^\star, s)\, g\left(\epsilon\, R\, a(s)\right)\, ds; \tag{2.10}$$

here $\epsilon > 0$ is any constant (choose and fix it) so that $1 - \frac{\mu M C}{R} \geq \epsilon$ (note ϵ exists since $R > r \geq \mu\, M\, C$). Then (2.1) has a nonnegative solution $y \in C[0,1]$ with $y(t) > 0$ for a.e. $t \in [0,1]$ (in fact $y(t) > 0$ at those $t's$ where $a(t) > 0$).