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ON THE CONFLUENT HYPERGEOMETRIC FUNCTIONS IN 2 VARIABLES

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§0. Introduction

Let $\lambda = (\lambda_0, \ldots, \lambda_{l-1})$ be a partition of n, sometimes called a '*Young diagram* λ' of weight n. Let $H_\lambda = J(\lambda_0) \times \cdots \times J(\lambda_{l-1}) \subset GL(n)$ be the associated maximal abelian subgroup with respect to λ, where $J(m)$ is the Jordan group of size m, i.e.,

$$J(m) = \left\{ \sum_{i=0}^{m-1} h_i \Lambda^i \mid h_0 \in \mathbb{C}^\times, h_1, \ldots, h_{m-1} \in \mathbb{C} \right\},$$

where the $m \times m$ matrix Λ is defined as

$$\Lambda = \begin{pmatrix} 0 & 1 & & \text{\huge 0} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \text{\huge 0} & & & 0 \end{pmatrix}.$$

We define the biholomorphic map

$$\begin{aligned} \iota : H_\lambda &\longrightarrow \prod_i (\mathbb{C}^\times \times \mathbb{C}^{\lambda_i - 1}) \\ h &\longmapsto (h_0^{(0)}, \ldots, h_{\lambda_0 - 1}^{(0)}, \ldots, h_0^{(l-1)}, \ldots, h_{\lambda_{l-1}-1}^{(l-1)}) \end{aligned}$$

where $h = (h^{(0)}, \ldots, h^{(l-1)})$, $h^{(i)} = \sum_{0 \le k < \lambda_i} h_k^{(i)} \Lambda^k \in J(\lambda_i)$.

Let $\alpha = (\alpha^{(0)}, \dots, \alpha^{(l-1)}), \alpha^{(i)} := (\alpha_0^{(i)}, \dots, \alpha_{\lambda_i-1}^{(i)})$ $(0 \le i \le l-1)$ be an n-tuple of complex numbers satisfying $\sum_{i=0}^{l-1} \alpha_0^{(i)} = -r$. We define the character $\chi(\cdot\,;\,\alpha) : \widetilde{H}_\lambda \longrightarrow \mathbb{C}^\times$ of the universal covering group $\widetilde{H}_\lambda = \widetilde{J}(\lambda_0) \times \cdots \times \widetilde{J}(\lambda_{l-1})$ of H_λ by $\chi(h;\alpha) = \prod_{i=0}^{l-1} \chi(h^{(i)};\alpha^{(i)})$, where

$$\chi(h^{(i)};\alpha^{(i)}) = h_0^{\alpha_0^{(i)}} \exp\left[\sum_{j=1}^{\lambda_i-1} \alpha_j^{(i)} \theta_j\left(\frac{h_1^{(i)}}{h_0^{(i)}}, \dots, \frac{h_{\lambda_i-1}^{(i)}}{h_0^{(i)}}\right)\right]$$

where θ_j are defined as the coefficients of the generating series

$$\log(1 + x_1 T + x_2 T^2 + \cdots) = \sum_{j=0}^{\infty} \theta_j(x_1, \dots, x_j) T^j.$$

Recall that the hypergeometric function $\Phi(z;\alpha)$ of type λ (see [K-H-T]) is a function defined by

$$\Phi(z;\alpha) = \int_\Delta \chi(\iota^{-1}(tz);\alpha) \cdot \omega \quad \text{for} \quad z \in Z_{r,n} \tag{0.1}$$

where $Z_{r,n}$ is the set of $r \times n$ complex matrices *in general position* (see [K-H-T]) with respect to λ, $\omega := \sum_{0 \le i < r} (-1)^i t_i dt_0 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_{r-1}$ and Δ is a *twisted cycle* in the t-space depending on z and α. Note that for $\lambda = (1, \dots, 1)$, the hypergeometric functions of type λ coincide with the general hypergeometric function defined in [G].

The set $Z_{r,n}$ admits an action of the group $GL(r) \times H_\lambda$:

$$\begin{aligned} GL(r) \times Z_{r,n} \times H_\lambda &\longrightarrow Z_{r,n} \\ (g, z, h) &\longmapsto gzh, \end{aligned}$$

under which Φ behaves as

$$\Phi(gz;\alpha) = (\det g)^{-1} \Phi(z;\alpha) \qquad g \in GL(r) \tag{0.2}$$

$$\Phi(zh_\lambda;\alpha) = \chi(h_\lambda) \Phi(z;\alpha) \qquad h_\lambda \in H_\lambda. \tag{0.3}$$

Furthermore, the function Φ admits another symmetry:

$$\Phi(zw_\lambda;\alpha) = \Phi(z;\alpha^t w_\lambda) \qquad w_\lambda \in W_\lambda, \tag{0.4}$$