MOLECULES OF THE HARDY SPACE AND THE NAVIER–STOKES EQUATIONS

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Abstract: For the incompressible Navier–Stokes system, we consider data $\vec{u}_0 \in (\mathrm{L}^3)^3$ whose Laplacian is a molecule of the Hardy space. We prove that this property is preserved by the only solution in $C\left([0,T[, (\mathrm{L}^3(\mathbb{R}^3))^3\right)$ at least at the beginning of its evolution.

In this paper we analyze the action of the Navier-Stokes equations on a subspace of $C\left([0,T[, (\mathrm{L}^3(\mathbb{R}^3))^3\right)$ which is specially adapted to study the location and oscillation of the diffusion term $\Delta\vec{u}$. Let us recall the Navier-Stokes system describing the motion of an incompressible, homogeneous, viscous fluid filling out the whole space $\mathbb{R}^3$, without the action of external forces:

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}\cdot(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

where $\vec{u}(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the velocity vector field and $p(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the pressure.

Dealing with the Cauchy problem with initial data $\vec{u}(0,x) = \vec{u}_0(x)$, it is possible to rewrite the system in the following integral expression:

$$\vec{u}(t) = \mathrm{e}^{t\Delta}\vec{u}_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}\cdot(\vec{u}\otimes\vec{u})(s)ds \tag{1}$$

where $\mathrm{e}^{t\Delta}$ is the heat semigroup defined by the convolution with the Gauss kernel:

$$\mathrm{e}^{t\Delta}f(x) = \left(\frac{1}{(4\pi t)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|\cdot|^2}{4t}} * f\right)(x)$$

and $\mathbb{P}$ is the projection operator on the divergence–free vector fields, defined by the matrix:

$$\begin{bmatrix} \mathrm{Id} + R_1R_1 & R_1R_2 & R_1R_3 \\ R_2R_1 & \mathrm{Id} + R_2R_2 & R_2R_3 \\ R_3R_1 & R_3R_2 & \mathrm{Id} + R_3R_3 \end{bmatrix}$$

where $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$ are the Riesz transforms for $j = 1, 2, 3$.

In fact the integral and the differential systems are equivalent in this framework, as it is shown in [FLT] under slight assumptions verified by a large class of spaces.

Given a Banach space X, the problem of the existence and the uniqueness of a function $\vec{u}(t) \in C([0, T[, X^3)$ solution of (1) in $\mathcal{S}'$ (a so–called mild solution) is normally approached by a fixed point argument. It is now well known ([FLT], [MA], [LE2]) that for a given $\vec{u}_0 \in (\mathrm{L}^3(\mathbb{R}^3))^3$ the only solution in $C\left([0, T[, (\mathrm{L}^3(\mathbb{R}^3))^3\right)$ belongs to every $(\mathrm{L}^p(\mathbb{R}^3))^3$ space with $p > 3$ for all its existence time. We are now interested in investigating the following problem: if the data $\vec{u}_0$ have some extra properties, does the solution keep verifying them at least on a small interval of time?

In [FLT] it was already established that if $\Delta\vec{u}_0 \in (\mathcal{H}^1)^3$, the Hardy space, then the solution $\vec{u}(t)$ satisfies $\Delta\vec{u}(t) \in (\mathcal{H}^1)^3$ for $0 < t < T' \leq T$. We will consider here a particular case of the previous one where the Laplacian of the data is a "molecule" for the Hardy space, and so well-localized in the space variables; we will prove then that the solution verifies the same property, namely its diffusion term keeps on being well–localized at the beginning of the motion.

In the following the reference space will always be $\mathbb{R}^3$.

More precisely, let $\delta \in]\frac{3}{2}, \frac{9}{2}[$, $\delta \neq \frac{5}{2}, \frac{7}{2}$ and let us introduce the following functional space:

$$X_\delta = \left\{ u \in \mathcal{S}' \ : \begin{array}{ll} u \quad \text{vanishes at infinity}, & \\ \Delta u \in \mathrm{L}^2((1+|x|^{2\delta})dx), & \\ \int \Delta u(x)dx = 0 & \text{for } \ \frac{3}{2} < \delta < \frac{9}{2} \\ \int x_i \Delta u(x)dx = 0 & \text{for } \ \frac{5}{2} < \delta < \frac{9}{2}, \ i = 1,2,3 \\ \int x_i x_j \Delta u(x)dx = 0 & \text{for } \ \frac{7}{2} < \delta < \frac{9}{2}, \ i,j = 1,2,3 \end{array} \right\}.$$

The space X_δ normed by $\|u\|_{X_\delta} = \|\Delta u\|_{\mathrm{L}^2((1+|x|^{2\delta})dx)}$ is a Banach space.

The theorem we are going to prove is the following one.

Theorem .
Let $\vec{u}_0 \in X_\delta^3$ and $\vec{\nabla}\cdot\vec{u}_0 = 0$. Then the mild solution of the Navier–Stokes equations $\vec{u}(t) \in C([0, T[, (\mathrm{L}^3)^3)$ is also in $C\left([0, T'], {X_\delta}^3\right)$ for $0 < T' \leq T$.

We would like to stress that the solution we obtain decays pointwise in space like $\dfrac{1}{1+|x|^{\delta-\frac{1}{2}}}$ for every $t \in [0, T']$ (see also proposition 2). In the particular case $\frac{3}{2} < \delta < \frac{7}{2}$, this decay property may also be recovered by the results of Miyakawa in [MI1], who establishes a pointwise space–time asymptotic behavior with respect to space–time variables x and t for a particular class of data.

For other results close to these topics see also [AGSS], [B2], [C], [HX], [MI2], [SS], [T].

I. GENERAL PROPERTIES OF X_δ

In the following we will write

$$\mathrm{L}^2_\delta = \mathrm{L}^2((1+|x|^{2\delta})dx)$$
$$\|u\|_{\mathrm{L}^2_\delta} = \|u(x)\|_{\mathrm{L}^2((1+|x|^{2\delta})dx)},$$
$$\|u\|_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}} = \left\|(1+|x|^{\delta-\frac{1}{2}})u(x)\right\|_{\mathrm{L}^\infty}.$$

We remark that the definition of X_δ is well posed; indeed if $\Delta u \in \mathrm{L}^2_\delta$ then $\Delta u \in \mathrm{L}^1$, for $\delta > \frac{3}{2}$, $x_i\Delta u \in \mathrm{L}^1$ for $i = 1, 2, 3$ and $\delta > \frac{5}{2}$, $x_i x_j \Delta u \in \mathrm{L}^1$ for $i, j = 1, 2, 3$ and $\delta > \frac{7}{2}$. Moreover, $X_\delta \subset \Delta\mathcal{H}^1$ and more precisely