Funkcialaj Ekvacioj
Volume 45, No. 1, pp 123--140

Collapsible Backward Continuation and Oscillations in Retarded Differential Equations

†L.A.V. Carvalho [1] ‡K.L.Cooke[2] and § L.A.C. Ladeira

†Departamento de Matemática, Universidade Estadual de Maringá
Maringá, PR, CEP 87020-900, Brazil
‡Department of Mathematics, Pomona College, Claremont, CA 91711
§Departamento de Matemática, ICMSC-USP
São Carlos, SP, CEP 13560-970, Brazil

AMS subject classifications: 34K15, 34K20.
KEYWORDS: oscillation, backwards oscillation, differential-difference equation, retarded differential equation, collapsible backward continuation.

1 Introduction.

We are concerned here with the oscillatory character of certain solutions of scalar differential equations of the form

$$\dot{x}(t) = f(t, x(t), x(t - r(t))), \tag{1}$$

where $t - r(t)$ is a given strictly increasing map defined for $t \geq t_o$ for some $t_o \in \mathbb{R}$. We also assume that $r(t_o) = 0$ and $0 < r(t) \leq t - t_o$ for $t \geq t_o$. Eq. (1) is a particular form of a "retarded differential equation", the delay in time being provided by the argument $t - r(t)$ and it is a special kind of a functional differential equation of the type

$$\dot{x}(t) = g(t, x_t), \tag{2}$$

where x_t is the "tail" map (as it is given, for instance, in [6]), $x_t(\theta) = x(t + \theta)$, $\theta \in [-r(t), 0]$. A particular instance of the more general situation to be investigated in this paper, was studied in [2], namely, the "scaled differential equation"

$$\dot{x}(t) = -ax(t) + ax(pt), \tag{3}$$

[1]Partially supported by CNPq, Proc. 304041/85-8
[2]Partially supported by NSF Grant DMS 9502922

where $a > 0$ and $0 < p < 1$ are given real numbers. Here, we have $t_o = 0$ and $r(t) = t - pt$. The nomenclature for Eq. (3) is due to the change of scale in time of the argument pt. Variations of this equation have been used in some mathematical models for pantograph equipment ([3, 4, 5, 9, 10, 13, 14, 15]). Eq. (1) has an interesting feature: two different kinds of initial value problem (IVP) can be attached to it, namely, the punctual IVP "$x(t_o) = x_o$," or the functional IVP "$x(t) = \psi(t)$", where x_o is arbitrarily chosen in $\mathbb{R}$ or ψ is arbitrarily chosen in $\mathcal{C}_\tau =: \mathcal{C}([\tau', \tau], \mathbb{R})$, and $\tau > t_o$ is an arbitrary real constant, with $\tau' = \tau - r(\tau)$. Here, as usual, $\mathcal{C}([a, b], \mathbb{R})$ stands for the space of the continuous maps from $[a, b]$ into $\mathbb{R}$, equipped with the supremum norm $||\phi|| = \sup\{|\phi(t)| : t \in [a, b]\}$. One often refers to either (t_o, x_o) or (τ, ψ) as an "initial condition". This duality of IVPs directly leads to the phenomenon of collapse of backward continuation ([2]), responsible for a wild kind of oscillatory behavior of solutions of Eq. (1), as we shall see. In order to make notations simpler, we extend the above nomenclature of τ' by putting $\tau'' = (\tau')' = \tau' - r(\tau')$ and so on.

By a "solution" of the punctual IVP we mean a **differentiable** map $x(t)$, defined in an interval $[t_o, b]$ which, of course, satisfies Eq. (1) for $t \in [t_o, b]$ and $x(t_o) = x_o$. The derivative at t_o means, of course, the right hand derivative while the derivative at b is the left hand one. As usual, the notations $\dot{x}(t+)$ and $\dot{x}(t-)$ denote the right-hand and the left-hand derivative of x at t, respectively. Similarly, a "solution" of the functional IVP is a **continuous** map $x : [\tau', b] \to \mathbb{R}$, such that $b > \tau$, x is **differentiable** in $[\tau, b]$ and, of course, satisfies $x(t) = \psi(t)$, $t \in [\tau', \tau]$, $\dot{x}(t) = f(t, x(t), x(t - r(t))$, $t \in [\tau, b]$ with the derivative at τ being the right-hand one, etc.. To emphasize the dependence on τ and ψ of this solution we shall denote it by "$x(\,.\,, \tau, \psi)$." The solution of the punctual IVP will be denoted simply by "$x(\,.\,, x_o)$." We assume from now the following conditions:

Hypothesis (H): **(i)**- $f(t, x, y)$ is continuous in t and C^∞ with respect to x, y for $t \geq t_o$ (this condition implies that $f(t, \,.\,, \,.\,)$ is locally Lipschitz, i.e., for each compact rectangle $Q = [a, b] \times [c, d]$ and $t \in \mathbb{R}$ there is a constant $M_t^Q > 0$ such that

$$|f(t, x, y) - f(t, u, v)| \leq M_t^Q \max\{|x - u|, |y - v| : (x, y), (u, v) \in Q\},$$

where $|.|$ denotes the modulus or absolute value map); **(ii)**- $D_y f(t, x, y) \neq 0$ for all (t, x, y), $t \geq t_o$ ((ii) implies that $f(t, x, \,.\,)$ is strictly monotone);