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Domains of Attraction of Competition-Diffusion Systems

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1. Introduction

In this paper we study the global behavior of solutions to the *reaction-diffusion system* :

$$(1.1)\qquad \begin{cases} u_t = d_1 \Delta u + u f(u,v) & \text{in } \Omega \times (0,\infty), \\ v_t = d_2 \Delta v + v g(u,v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and ν is the outward unit normal vector to $\partial\Omega$. The initial functions $u_0(x)$, $v_0(x)$ are not identically zero, and the functions $f(u,v)$ and $g(u,v)$ are of class $C^1(Q)$ where

$$Q = \{(u,v) \in \mathbb{R}^2 | u \geq 0, v \geq 0\}.$$

For $f(u,v)$ and $g(u,v)$ we will consider the following two cases of *competition type* which make (1.1) a *competition-diffusion system* :

(α) $f(u,v) = a_1 - b_1 u - c_1 v, \qquad g(u,v) = a_2 - b_2 u - c_2 v,$

(β) $f(u,v) = a_1 - b_1 u^2 - c_1 v^2, \quad g(u,v) = a_2 - b_2 u^2 - c_2 v^2.$

In the system (1.1) u and v are nonnegative functions which represent the population densities of two competing species. d_1 and d_2 are the *diffusion* rates of the two species, respectively. a_1 and a_2 denote the intrinsic growth rates, b_1 and c_2 account for intra-specific competitions, b_2 and c_1 are the coefficients for inter-specific competitions. For details on the backgrounds of this model, we refer the reader to [6].

Remark. The linear functions for $f(u,v)$ and $g(u,v)$ as in (α) are often used in the classical *competition-diffusion systems.* Though the quadratic functions for $f(u,v)$ and $g(u,v)$ as in (β) may not be used commonly, they make the system (1.1) the gradient system of an energy functional (after simple scalings) which helps one to analyze the system (1.1) more clearly. And, in the course of this paper it will be shown that the system (1.1) with (β) has similar properties as the system with (α).

The global behavior of solutions to the system (1.1) is related to that of its *kinetic system* which is the following system of ordinary differential equations :

$$\left\{ \begin{array}{ll} u_t = uf(u,v) & \text{in } (0,\infty), \\ v_t = vg(u,v) & \text{in } (0,\infty). \end{array} \right. \tag{1.2}$$

Clearly Q is positively invariant for the flow of (1.2). The equilibria of the kinetic system (1.2) in Q consist of four points $(u_A, 0)$, $(0, v_B)$, (u_C, v_C) and $(0,0)$, where u_A, v_B, u_C and v_C are positive constants in both cases (α) and (β) of the functions $f(u,v)$ and $g(u,v)$. Throughout this paper we impose the following *strong competition* conditions on the coefficients in $f(u,v)$ and $g(u,v)$:

$$\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}. \tag{1.3}$$

We note that the condition (1.3) assures that $(0,0)$ is unstable, $(u_A, 0)$, $(0, v_B)$ are stable, and (u_C, v_C) is a saddle point for (1.2). The flows of the kinetic system (1.2) under the condition (1.3) are shown in Figure 1.

For the general properties of separatrix $h(u)$ of the kinetic system (1.2) which is illustrated in Figure 1 we refer the reader to the result due to Iida et al. [2] which is stated in Proposition 1.1 in the following. The reader may also refer to Hirsch and Smale [1], Ninomiya [5] for the properties of separatrices.

Proposition 1.1. *Suppose for the system* (1.2) *that* $f(u,v)$ *and* $g(u,v)$ *are as in either* (α) *or* (β). *Then there exists a monotone function* $h(u)$ *defined on* $[0, u_\infty)$ *with* $u_\infty \in (u_C, \infty]$ *such that*

$W_A = \{(u,v) \in Q | v < h(u)\}$ *is the basin of attraction for* $(u_A, 0)$,