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Remarks on viscosity solutions of the Dirichlet problem for quasilinear degenerate elliptic equations

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1. Introduction.

In this paper we are concerned with the Dirichlet problem (hereafter called (**DP**)) for the quasilinear degenerate elliptic equation :

$$-g(|x|, u)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad B_R \tag{1.1}$$

$$u = \beta \quad \text{on} \quad \partial B_R, \tag{1.2}$$

where $B_R = \{x \in \mathbf{R}^N; |x| < R\}, N \geq 2, g : [0, R] \times \mathbf{R} \to \mathbf{R}^+ = [0, \infty)$ is a given continuous function, Δ is the Laplacian, and β is a real number such that $f(R, \beta) = 0$.

This investigation is a sequel of our previous work [3] where we studied the existence, uniqueness, nonuniqueness and radial property of viscosity solutions of the Dirichlet problem for the semilinear degenerate elliptic equation

$$-g(|x|)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad B_R, \tag{1.3}$$

where g is a nonnegative continuous function. We refer the reader to the Monograph by Crandall, Ishii and Lions [1] for definitions, details and references of viscosity solutions.

The main purpose of the present paper is to prove existence of viscosity solutions, and to give a sufficient condition assuring the uniqueness and the radial

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symmetry of viscosity solutions of (**DP**). In what follows we consider the problem (**DP**) in case $N = 2$, since we can treat it in case $N \geq 3$ by the same arguments.

Throughout this paper we make the following assumptions:

(H1) $f(t, y) \in C([0, R] \times \mathbf{R})$ is strictly increasing in y for each fixed $t \in [0, R]$.

(H2) There exists an implicit function $\varphi(t)$ of $f(t, y) = 0$ satisfying

$$\sup_{0 \leq s \leq R, s \neq t} \left| \frac{\varphi(s) - \varphi(t)}{s - t} \right| = \Psi(t) \in L^1(0, R).$$

It is clear that $\varphi(t)$ is continuous on $[0, R]$ by (H1) and (H2).

We state our existence theorem.

Theorem 1. *Under the assumptions* (H1) *and* (H2) *there exists a radial viscosity solution of* (**DP**).

In order to establish the uniqueness of viscosity solutions for (**DP**) we introduce additional assumptions and a notion of standard viscosity solution.

(H3) $\Psi(t) \in L^\infty(0, R)$, here Ψ is the function defined in (H2).

(H4) The function g satisfies the condition : if $g(t_1, y_1) = 0$ then

$$\begin{cases} g(s, y_1) \leq \text{Const.} \mid s - t_1 \mid^2 & \text{for} \quad \forall s \in N(t_1), \\ g(s, y) \leq \text{Const.}(\mid s - t_1 \mid + \mid y - y_1 \mid) & \text{for} \quad \forall (s, y) \in N(t_1, y_1), \end{cases}$$

where $N(t_1)$ and $N(t_1, y_1)$ are small neighborhoods of t_1 and (t_1, y_1), respectively.

(H5) For f and g, we impose the following structure condition : if $0 \leq t \leq R$, $y_1 < y_2$ and $g(t, y_1) + g(t, y_2) > 0$, then

$$g(t, y_1) f(t, y_2) - g(t, y_2) f(t, y_1) > 0.$$

Example Let $a \in C^2([0, R]), b \in C^2([0, R]; [0, 1])$. Define $g \in C^2([0, R] \times \mathbf{R})$ by

$$g(t, y) = 1 - \cos[h(y - a(t); b(t))]$$