Funkcialaj Ekvacioj
Volume 45, No. 1, pp 71--88

Nonoscillatory Solutions of Fourth Order Quasilinear Differential Equations

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1. Introduction

This paper is concerned with the oscillatory and nonoscillatory behavior of solutions of fourth order quasilinear differential equations of the form

$$(p(t)|u''|^{\alpha-1}u'')'' + q(t)|u|^{\beta-1}u = 0, \tag{1.1}$$

where α, β are positive constants and $p(t)$, $q(t)$ are positive continuous functions defined on $[a, \infty)$, $a > 0$. We assume that $p(t)$ satisfies

$$\int_a^\infty \left[\frac{t}{p(t)}\right]^{1/\alpha} dt = \infty, \tag{1.2}$$

or, more strongly,

$$\int_a^\infty \frac{t}{[p(t)]^{1/\alpha}} dt = \infty \quad \text{and} \quad \int_a^\infty \left[\frac{t}{p(t)}\right]^{1/\alpha} dt = \infty. \tag{1.3}$$

By a solution of (1.1) we mean a real-valued function $u(t)$ such that $u \in C^2[b,\infty)$ and $p|u''|^{\alpha-1}u'' \in C^2[b,\infty)$ and $u(t)$ satisfies (1.1) at every point of $[b,\infty)$, where $b \geq a$ and b may depend on $u(t)$. Such a solution $u(t)$ of (1.1) is called nonoscillatory if $u(t)$ is eventually positive or eventually negative. A solution $u(t)$ of (1.1) is called oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$. Equation (1.1) itself is called oscillatory if all of its solutions are oscillatory.

The main objective is to investigate the oscillatory and nonoscillatory behavior of solutions of (1.1). We first study the structure of the set of nonoscillatory solutions of (1.1). It is observed that a solution $u(t)$ which is asymptotic to a positive constant as $t \to \infty$ is "minimal" in the set of all eventually positive solutions of (1.1), and a solution $u(t)$ which is asymptotic to a positive constant multiple of the function

$$\int_a^t (t-s)\left[\frac{s}{p(s)}\right]^{1/\alpha} ds$$

as $t \to \infty$ is "maximal" in the set of all eventually positive solutions of (1.1). We establish the necessary and sufficient conditions for the existence of "minimal" and "maximal" solutions of (1.1). These necessary and sufficient conditions are given by certain integral conditions on $p(t)$ and $q(t)$. Under the assumptions $\alpha \geq 1 > \beta$ and $\alpha \leq 1 < \beta$, we can present the necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1). In the case of $\alpha \geq 1 > \beta$ [resp. $\alpha \leq 1 < \beta$], the necessary and sufficient condition is identical to the integral condition which characterizes the existence of maximal [resp. minimal] solutions.

In the case $\alpha = 1$, equation (1.1) is

$$(p(t)u'')'' + q(t)|u|^{\beta-1}u = 0, \tag{1.4}$$

and both of conditions (1.2) and (1.3) are

$$\int_a^\infty \frac{t}{p(t)}\, dt = \infty. \tag{1.5}$$

The oscillatory and nonoscillatory behavior of solutions of (1.4) under the condition (1.5) has been studied by Kusano and Naito [4]. The results of the present paper generalize those of [4].

Now, consider the second order quasilinear differential equation

$$(p(t)|u'|^{\alpha-1}u')' + q(t)|u|^{\beta-1}u = 0, \tag{1.6}$$

where $\alpha > 0$, $\beta > 0$, and $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, $a > 0$. Suppose that

$$\int_a^\infty \frac{dt}{[p(t)]^{1/\alpha}} = \infty.$$