Funkcialaj Ekvacioj
Volume 45, No. 1, pp 53--69

SHARP ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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1. Introduction

We study the following nonlinear Schrödinger equation in one space dimension:

$$\begin{aligned} iu_t(t,x) &= (-1/2)\partial^2 u(t,x) + f(u(t,x)), \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.1}$$

Here $(t,x) \in \mathbb{R}\times\mathbb{R}$, $\partial = \partial/\partial x$ and $f(u) = f_1(u) + f_2(u) = \lambda_1 |u|^{p_1-1}u + \lambda_2 |u|^{p_2-1}u$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $1 < p_1 < p_2$. The aim of this paper is to study the asymptotic behavior of the solution as $t \to \infty$, especially to obtain the second term of the asymptotic expansion of the solution in the case $p_1 = 3$.

There is a large literature on the equation (1.1); see [1, 3–9, 11, 13–23] and references therein. The well-posedness of the Cauchy Problem (1.1) has been extensively studied, and the results already obtained are satisfactory for our study of the asymptotic behavior of the solution. To put it briefly, if $1 \le p_1 < p_2 < 5$, the equation (1.1) is (conditionally) well-posed in L^2, and moreover $U(-t)u(t) \in L^{2,s}$ if $u_0 \in L^{2,s}$ with $0 < s < p_1$. Here $U(t) = \exp(it\partial^2/2)$ is the free propagator and $L^{2,s}$ denotes the weighted L^2-space of order s; more precisely, see Proposition 2.4 below. In what follows, we simply call the solution obtained by Proposition 2.4 "the solution to (1.1)."

The asymptotic behavior of the solution to (1.1) is usually explained in terms of the scattering theory. When $t \to \infty$, the solution is expected to decay by the dispersive effect of the equation. Hence we can expect

*Supported by JSPS Research Fellowships for Young Scientists.

**Supported in part by the Yukawa foundation of Osaka University.

that the nonlinearity in the equation decays rapidly enough and loses its effect as $t \to \infty$. Thus the expected profile of the solution to (1.1) is of the form $U(t)\phi$, which is a solution to the free Schrödinger equation; here ϕ is a suitable function called the scattering state of the solution. This observation is, however, correct only in case $3 < p_1 < p_2$, namely the short-range case (on the other hand, the nonlinear term with $p_1 \le 3$ is called of long-range). Indeed, the followings are well known [1, 12, 23].

(I) If $\lambda_1, \lambda_2 \ge 0$, $3 < p_1 < p_2 < 5$ and $u_0 \in L^{2,1}$, then there exists a function $\phi \in L^2$ satisfying

$$\lim_{t\to\infty} \|u(t) - U(t)\phi\|_2 = 0, \tag{1.2}$$

where $u(t)$ is the solution to (1.1).

(II) If $3 < p_1 < p_2 < 5$, $u_0 \in L^{2,1}$ and $\|u_0\|_{L^{2,1}}$ is sufficiently small, then there exists a function $\phi \in L^{2,1}$ satisfying

$$\lim_{t\to\infty} \|U(-t)u(t) - \phi\|_{L^{2,1}} = 0, \tag{1.3}$$

where $u(t)$ is the solution to (1.1).

On the other hand, we have

(III) If $\lambda_1 \neq 0$, $p_1 \le 3$ and $u_0 \in L^{2,1} \setminus \{0\}$, then there does not exist a function $\phi \in L^2$ satisfying (1.2) for the solution $u(t)$ to (1.1).

From the results above, the critical exponent for the existence of the scattering state is $p_1 = 3$. In this case there does not exist usual scattering state, but if we introduce the modified free dynamics of the form $U(t)\exp\bigl(-iS(t,-i\partial)\bigr)\phi$, the situation is improved, where $S(t,\xi) = \lambda_1|\hat{\phi}(\xi)|^2 \log t$ is the modifier of the Dollard type. Indeed, the following is known [11].

(IV) If $3 = p_1 < p_2 < 5$, $u_0 \in L^{2,s}$ for some $s > 1/2$, and $\|u_0\|_{L^{2,s}}$ is sufficiently small, then there exists a function $\phi \in L^2 \cap L^\infty$ satisfying

$$\lim_{t\to\infty} \|u(t) - U(t)\exp\bigl(-iS(t,-i\partial)\bigr)\phi\|_2 = 0, \tag{1.4}$$

where $u(t)$ is the solution to (1.1) and $S(t,\xi)$ is defined as above.