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On L^p Regularity for Weak Derivatives of Spherically Symmetric Solutions of the Porous Media Equation

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1. Introduction. We consider the Cauchy problem for the porous media equation

$$(1.1) \qquad \begin{cases} u_t = \Delta u^m & \text{in} \quad R^N \times (0,T) \\ u(x,0) = u_0 & \text{on} \quad R^N, \end{cases}$$

where $m > 1$ and $u_0 \geq 0$. By a solution of (1.1) we mean a function $u(x,t)$ such that

$$u(x,t) \geq 0 \text{ in } R^N \times (0,T),$$

$$\int_0^T \int_{R^N} \left[u(x,t)^2 + |\nabla_x u(x,t)|^2 \right] dxdt < \infty$$

and

$$\int_0^T \int_{R^N} (u\phi_t - \nabla_x u^m \cdot \nabla_x \phi)\, dxdt + \int_{R^N} u_0(x)\phi(x,0)dx = 0$$

for any continuously differentiable function $\phi(x,t)$ with compact support in $R^N \times [0,T)$. The problem(1.1) has been studied by many authors. For a detailed account of (1.1) we refer to the survey of Kalashnikov [6]. The existence and the uniqueness of solutions of (1.1) are due to [8] and [9] under some assumption on u_0.

We are concerned to the regularity property for u. The local Hölder continuity of u was shown by Caffarelli and Friedman [3]. Aronson and

Bénilan [1] proved that Δu^m belongs to $L^1_{loc}(R^N \times (0,T))$. The method of their proof is to obtain the inequarity

$$u_t \geq -\frac{1}{t}\left(m-1+\frac{2}{N}\right)^{-1} \quad in \;\; R^N \times (0,T),$$

which is in the distribution sence. Soon after Bénilan [2] proved that

$$\Delta u^m \in L^p_{loc}(R^N \times (0,T)),$$

if $1<p<1+\frac{1}{m}$. Here p needs to be less than 2 in virtue of $m>1$. When $p=2$, there is the following result by one of the authors [4]:

$$\partial_{x_i}\partial_{x_j} u^m \in L^2(R^N \times (0,T)), \qquad i,j=1,\cdots,N,$$

if $1<m<\frac{3N}{3N-2}$. When u is spherically symmetric under some assumption on u_0, this result was improved in [5] as follows:

the condition $1<m<\frac{3N}{3N-2}$ *can be weakend with* $1<m<3$, *for* $N=1,2,3$.

We consider the well-known Barenblatt solution:

$$w(x,t)=(t+\tau)^{-k}\left(\left[a^2-\frac{k(m-1)}{2Nm}\frac{|x|^2}{(t+\tau)^{\frac{2k}{N}}}\right]_+\right)^{\frac{1}{m-1}},$$

where $a,\ \tau>0$ and $k=\left(m-\frac{N-2}{N}\right)^{-1}$. We rewrite w simply with $(t+\tau)^{-k}\left([g]_+\right)^{\frac{1}{m-1}}$. Then obviously

$$\partial_{x_i}\partial_{x_j} w^m=\left([g]_+\right)^{\frac{2-m}{m-1}}P_{ij}+\left([g]_+\right)^{\frac{1}{m-1}}Q_{ij},$$

where P_{ij} and Q_{ij} are smooth functions. Hence we see that

$$\partial_{x_i}\partial_{x_j} w^m \in L^p(R^N \times (0,T)), \;\text{ if }\; p\left(\frac{2-m}{m-1}\right)>-1.$$

Let u be the spherically symmetric solution of (1.1). Then from the above we conjecture that for any given $p>2$

$$\partial_{x_i}\partial_{x_j} u^m \in L^p(R^N \times (0,T)), \quad i,j=1,\cdots,N,$$