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Structure of positive radial solutions for semilinear Dirichlet problems on a ball

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1 Introduction

The structure of positive radial solutions for nonlinear elliptic equations have attracted much attention for these years. In particular, many interesting and beautiful results have been obtained concerning the structure of positive radial solutions on entire space $\mathbf{R}^n$ (see, e.g., the survey paper by Ni [8]). However, it is not straightforward to extend these results to boundary value problems on bounded domains.

In this paper, we consider the structure of solutions of the semilinear elliptic equation

$$\Delta u + Q(|x|)u^p = 0 \quad \text{in } B, \tag{1.1}$$

where $p > 1$,

$$u^p = \begin{cases} |u|^p & \text{if } u > 0, \\ 0 & \text{if } u \le 0, \end{cases}$$

$Q(|x|)$ is a given nonnegative function, and

$$B = \{x \in \mathbf{R}^n; |x| < 1\}, \quad n > 2.$$

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Our main concern is the existence and uniqueness of positive radial solutions of (1.1) under the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial B. \tag{1.2}$$

Before studying the boundary value problem, we describe the result of [11] concerning the structure of positive radial solutions of (1.1) in the entire space $\mathbf{R}^n$. Since we are concerned with positive radial solutions, we consider the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + K(r)u^p = 0, & r \in (0, \infty), \\ u(0) = \alpha > 0. \end{cases} \tag{1.3}$$

Here we assume that $p > 1$, $n > 2$, and $K(r)$ satisfies

$$\text{(K)} \quad \begin{cases} K(r) \in C((0, \infty)); \\ K(r) \geq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0, \infty); \\ rK(r) \in L^1(0, 1); \\ r^{n-1-(n-2)p} K(r) \in L^1(1, \infty). \end{cases}$$

We note that $K(r)$ may be unbounded at $r = 0$. Under the first and second conditions, it is shown in [7, 9] that the initial value problem (1.3) is uniquely solvable if and only if $rK(r) \in L^1(0, 1)$. We denote the unique solution by $u(r; \alpha)$. It is known [11] that the solution of (1.3) is classified as

(i) a crossing solution: $u(r; \alpha)$ has a zero in $(0, \infty)$,

(ii) a slowly decaying solution: $u(r; \alpha) > 0$ on $[0, \infty)$ and $r^{n-2}u(r; \alpha) \to \infty$ as $r \to \infty$,

(iii) a rapidly decaying solution: $u(r; \alpha) > 0$ on $[0, \infty)$ and $\lim_{r\to\infty} r^{n-2}u(r; \alpha)$ exists and is positive.

Finally, it is known [1, 6] that if $r^{n-1-(n-2)p} K(r) \notin L^1(1, \infty)$, then any solution of (1.3) (whether or not it satisfies the initial condition) cannot be positive near ∞ so that $u(r; \alpha)$ is a crossing solution for any $\alpha \in (0, \infty)$.