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Energy Decay of Solutions for the Semilinear Dissipative Wave Equations in an Exterior Domain

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Abstract

Uniform energy decay of solutions for the semilinear wave equations with a linear dissipation will be given to the exterior mixed problems. In order to derive the total energy decay property of a solution, an useful inequality due to Ikehata-Matsuyama [3] will be used. In fact, we shall derive the decay rate such as $(1+t)^2E(t) \leq C$ for small initial datum with the compact support, where $E(t)$ represents the total energy.

1 Introduction

Let $\Omega \subset R^N (N \geq 2)$ be an exterior domain with smooth compact boundary $\partial\Omega$. Without loss of generality we may assume $0 \notin \bar{\Omega}$. In this paper we are concerned with the initial-boundary value problem for the semilinear dissipative wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = |u(t,x)|^{p-1}u(t,x), \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.3}$$

Throughout this paper, $\|\cdot\|_q$ means the usual $L^q(\Omega)$-norm and in particular, we set $\|\cdot\| = \|\cdot\|_2$. Furthermore, we adopt

$$(f,g) = \int_\Omega f(x)g(x)dx$$

as the usual $L^2(\Omega)$-inner product. The total energy $E(t)$ to the equation (1.1) is defined by

$$E(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|^2.$$

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The main purpose of this paper is to derive a certain decay rate for the total energy $E(t)$ and L^2-norm of the solution $u(t,x)$ to the problem (1.1)-(1.3) with compact support initial data in an "exterior domain" through the multiplier method together with the semigroup theory. Our argument is based on the results due to Ikehata-Matsuyama [3] and Saeki-Ikehata [10] which derive the sharp decay estimates of the various norms of the solutions to the linear equation:

$$u_{tt} - \Delta u + u_t = 0.$$

For the related result, Nakao-Ono [9] studied the global solvability and energy decay to the Cauchy problem (1.1)-(1.2) with $\Omega = R^N$ through the modified potential-well method. Roughly speaking, they have derived the following results: let $1 + \dfrac{4}{N} \leq p < \dfrac{N+2}{[N-2]^+}$. Then, for small initial data $\|u_0\|_{H^1} + \|u_1\| \ll 1$ the Cauchy problem (1.1)-(1.2) with $\Omega = R^N$ has a global solution $u \in C([0,+\infty); H^1(R^N)) \cap C^1([0,+\infty); L^2(R^N))$ satisfying

$$\|u(t,\cdot)\|^2 \leq C, \quad E(t) \leq C(1+t)^{-1}.$$

For the present, it seems unknown whether the total energy $E(t)^{1/2}$ and more L^2-norm of a solution to the problem (1.1)-(1.3) in exterior domains decay faster than $(1+t)^{-1}$ or not. Our device is in the fact that we need not go through any so called the spectral analysis in order to obtain the decay rate as in Dan-Shibata [1].

Now before stating our main theorem we shall define a function $d(x)$ as follows:

$$d(x) = \begin{cases} |x| & N \geq 3, \\ |x|\log(B|x|) & N = 2, \end{cases} \tag{1.4}$$

where $B > 0$ is a constant such that $\inf_{x\in\Omega}|x| \geq \dfrac{2}{B} > 0$. We make some assumptions before introducing the main theorem.

(A.1) $1 + \dfrac{6}{N+2} < p \leq \dfrac{N}{N-2} \quad (N = 3)$,

(A.2) $1 + \dfrac{6}{N+2} < p < +\infty \quad (N = 2)$.

Let $\rho > 0$ be a real number such that $\partial\Omega \subset B_\rho$. Our final assumption is as follows: for each fixed $R > \rho$,

(A.3) $\mathit{supp}\, u_0 \cup \mathit{supp}\, u_1 \subset \Omega \cap B_R$.

Here $B_r = \{x \in R^N : |x| < r\}$. Further, set

$$I_0 = \|u_0\|_{H^1} + \|u_1\| + \|d(\cdot)(u_0 + u_1)\|.$$

Our result reads as follows:

Theorem 1.1 *Let $N = 3$. Under the assumptions* (A.1) *and* (A.3), *there exists a real number $\delta > 0$ such that if the initial data further satisfies $I_0 < \delta$, then the problem* (1.1)-(1.3) *has a global solution $u \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ satisfying*

$$E(t) \leq CI_0^2(1+t)^{-2}, \quad \|u(t,\cdot)\|^2 \leq CI_0^2(1+t)^{-1}.$$