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ANALYTICAL SIMPLIFICATION OF A NONLINEAR 2-SYSTEM WITH SIMPLIFIED EQUATIONS INVOLVING INFINITELY MANY TERMS

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To the memory of my brothers

§1. Introduction.

In a previous paper (M. Iwano [5]), the author studies a nonlinear 2-system of the form

$$\text{(A)} \qquad x^2\frac{dy}{dx} = (\mu + \alpha x)y + f(x,y,z), \quad x^2\frac{dz}{dx} = (-\nu + \beta x)z + g(x,y,z),$$

under the assumptions that

(i) *μ and ν are commensurable positive numbers.*

(ii) *α and β are real constants such that* $\alpha\mu + \beta\nu > 0$.

(iii) *$f(x,y,z)$ and $g(x,y,z)$ are holomorphic and bounded functions of (x,y,z) for $|x| < a, \quad |y| < b, \quad |z| < b$ and their Taylor series expansions in (x,y,z) involve neither the independent terms nor the linear terms with respect to y and z:*

$$\text{(1.1)} \qquad f(x,y,z) = \sum_{j+k\geq 2, i\geq 0} a_{i,j,k}x^iy^jz^k, \quad g(x,y,z) = \sum_{j+k\geq 2, i\geq 0} b_{i,j,k}x^iy^jz^k,$$

a and b being small positive constants.

To simplify the description, we assume that

$$\text{(1.2)} \qquad \mu = \nu = 1.$$

By virtue of the commensurability condition between $\mu(= 1)$ and $\nu(= 1)$, there remain infinitely many terms in the simplified equations, to which we can give any analytical meaning without applying a well known theorem named Borel-Ritt Theorem(for example, M. Hukuhara [1]). By applying a transformation of cubic polynomials in (x,u,v), we can reduce the values of $a_{0,j,k}$ for $j + k \leq 3$ to zero except for $(j,k) = (2,1)$ and the values of $b_{0,j,k}$ for $j + k \leq 3$ to zero except for $(j,k) = (1,2)$. It is already shown that, after such a transformation has been applied, the new values $a_{0,2,1} = \alpha'$ and $b_{0,1,2} = \beta'$ are given by

$$\text{(1.3)} \qquad \begin{cases} \alpha' = -a_{0,2,0}a_{0,1,1} + a_{0,1,1}b_{0,1,1} + \dfrac{2}{3}a_{0,0,2}b_{0,2,0} + a_{0,2,1}, \\ \beta' = b_{0,0,2}b_{0,1,1} - b_{0,1,1}a_{0,1,1} - \dfrac{2}{3}b_{0,2,0}a_{0,0,2} + b_{0,1,2}. \end{cases}$$

By assuming that

(iv) *the sum δ of the quantities α' and β' is a nonzero quantity* :

$$\delta \equiv \alpha' + \beta' \neq 0, \tag{1.4}$$

the author succeeds in constructing a formal transformation which reduces the f to *the product of y and a quadratic form in yz* and g to *the product of z and a quadratic form in yz*. The result is clarified in the theorem below:

Theorem A. *Apply successively two formal transformations $\{y, z\} \to \{u, v\}$ and $\{u, v\} \to \{\eta, \zeta\}$ of the types* :

$$y = u + \sum_{i\geq 0, j+k\geq 2} p_{i,j,k} x^i u^j v^k, \quad z = v + \sum_{i\geq 0, j+k\geq 2} q_{i,j,k} x^i u^j v^k \tag{1.5}$$

and

$$u = \eta\big(1 + \sum_{i+j\geq 1, j\geq 1} p_{i,j} x^i (\eta\zeta)^j\big), \quad v = \zeta\big(1 + \sum_{i+j\geq 1, j\geq 1} q_{i,j} x^i (\eta\zeta)^j\big). \tag{1.6}$$

Equations (A) *are formally changed to equations of the form*

$$\text{(B)} \quad \begin{cases} x^2 \dfrac{d\eta}{dx} &= \eta(1 + \alpha x + \alpha'\eta\zeta + \frac{\gamma}{\delta} x\alpha(x)\eta\zeta + \alpha(x)(\eta\zeta)^2), \\ x^2 \dfrac{d\zeta}{dx} &= \zeta(-1 + \beta x + \beta'\eta\zeta + \frac{\gamma}{\delta} x\beta(x)\eta\zeta + \beta(x)(\eta\zeta)^2). \end{cases}$$

The $\alpha(x)$ and $\beta(x)$ are expressed by power series in x. The equations (B) *are called the simplified equations.*

But, unfortunately, we cannot given any analytical meaning to the power series $\alpha(x)$ and $\beta(x)$ by a natural manner (which means that they are solutions of algebraic equations or solutions of simple differential equations.) So, by means of Borel-Ritt Theorem, we define the $\alpha(x)$ and $\beta(x)$ as *holomorphic functions* such that *they are holomorphic and bounded in x for a domain of the form*

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_0 \tag{1.7}$$

and, moreover, admit asymptotic expansions in powers of x as x tends to the origin through (1.7). The ϵ is a preassinged sufficiently small positive quantity and the a_0 is a small constant.

The purpose of the present paper is to discuss analytical meaning of the formal transformation $\{y, z\} \to \{\eta, \zeta\}$ of the following form, which are obtained by the composite of (1.5) and (1.6):

$$T : y = \eta + \sum_{j+k\geq 2} R_{j,k}(x)\eta^j\zeta^k, \quad z = \zeta + \sum_{j+k\geq 2} S_{j,k}(x)\eta^j\zeta^k. \tag{1.8}$$

The essential problem on giving analytical meaning to the formal transformation is the construction of stable domains of solutions of the simplified equations (B). The character of the stable domains depends on the sign of the quantity $\gamma = \alpha + \beta - 1$. As the consequence, there is an essential distinction between the case of $\gamma \neq 0$ and the case of $\gamma = 0$ as will be shown in the theorems below. We shall write down the main results.