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Differential equations for modular forms of level three

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1 Introduction

It is known that elliptic modular forms satisfy nonlinear differential equations of third order. Historically, it was Jacobi who first studied a differential equation whose solutions are written in terms of theta constants. He showed that the equation

$$\left(y^2y''' - 15yy'y'' + 30y'^3\right)^2 + 32\left(yy'' - 3y'^2\right)^3 = -\pi^2 y^{10}\left(yy'' - 3y'^2\right)^2,$$

has solutions $y = \theta_2, \theta_3, \theta_4$ ([7]). The study of Jacobi's equationis is complicated, but Halphen found a simple system of differential equations ([5])

$$\begin{cases} X' + Y' = 2XY, \\ Y' + Z' = 2YZ, \\ Z' + X' = 2ZX, \end{cases}$$

whose solutions are written in terms of logarithmic derivatives of theta constants ([5], [8]). Although Halphen's equations are not integrable in the classical sence (there exists no algebraic Hamiltonian), Halphen's equations can be solved exactly. Halphen's equations may be considered as fundamental equations for modular forms of level two. J. Chazy also found a differential equation for modular forms of level one ([2]):

$$y''' = 2yy'' - 3(y')^2. \tag{1}$$

The aim of this paper is to deduce Halphen type equations for modular forms of level three:

$$\begin{cases} W' + X' + Y' = WX + XY + YW, \\ W' + Y' + Z' = WY + YZ + ZW, \\ W' + X' + Z' = WX + XZ + ZW, \\ X' + Y' + Z' = XY + YZ + ZX, \\ e^{\frac{4}{3}\pi i}(XZ + YW) + e^{\frac{2}{3}\pi i}(XW + YZ) + (XY + ZW) = 0. \end{cases} \tag{2}$$

A special solution of (2) is given by

$$\begin{aligned} W &= 3\frac{\partial}{\partial\tau}\log\eta\left(\frac{\tau}{3}\right) - \frac{\partial}{\partial\tau}\log\eta(\tau), \\ X &= 3\frac{\partial}{\partial\tau}\log\eta\left(3\tau\right) - \frac{\partial}{\partial\tau}\log\eta(\tau), \\ Y &= 3\frac{\partial}{\partial\tau}\log\eta\left(\frac{\tau+2}{3}\right) - \frac{\partial}{\partial\tau}\log\eta(\tau), \\ Z &= 3\frac{\partial}{\partial\tau}\log\eta\left(\frac{\tau+1}{3}\right) - \frac{\partial}{\partial\tau}\log\eta(\tau). \end{aligned} \tag{3}$$

The system (2) is invariant under the following $SL(2,\mathbb{C})$-action.

$$\widetilde{X_j}(\tau) = \frac{1}{(c\tau+d)^2}X_j\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{c}{c\tau+d},$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. Since (2) is equivalent to an equation of third order, generic solutions are given by the $SL(2,\mathbb{C})$-orbit of (3). In section 4, we will study the whole solution space of (2). The solution space of (2) decomposes into one generic orbit and several special orbits. Any solution in special orbits is a rational function.

In section 2, we will study the Hesse pencil, following [1] and [3]. Since the Hesse pencil is an elliptic modular surface for the modular group of level three, the period of the Hesse pencil is a modular form of level three. In section 3, we will deduce Halphen type equations of level three from the Picard-Fuchs equation of the Hesse pencil. A method for constructing Halphen type equations was found by Jacobi ([7]) and is generalized in [9]. Recently this method has been applied to replicable functions by Harnad and McKay ([6]).

Halphen's equation is a specialization of self-dual Einstein equation ([4]) and other Halphen type equations also have many applications in mathematical physics ([9]). It would be interesting to find an application of (2) in mathematical physics. It seems that nonlinear differential equations defining modular forms are closely related to self-dual equations.

Chazy considered the equation (1) in his study on classification of Painlevé type equations of third order ([2]). Although there is no singularity in (1), solutions of (1) have natural boundaries. Other Halphen type equations also do not have the Painlevé property. Halphen type equations may be significant examples for considering the meaning of the Painlevé property.