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Oscillatory and Asymptotic Behavior of Solutions of Third Order Delay Difference Equations

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1. Introduction

In this paper, we study the oscillatory and nonoscillatory behavior of solutions of the delay difference equation

$$(1) \qquad \Delta(a_n\Delta(b_n\Delta y_n)) + q_n f(y_{n-m+1}) = h_n, \qquad n \in \boldsymbol{N}_0 = \{0, 1, 2, \dots\}$$

where $\{a_n\}, \{b_n\}, \{q_n\}$, and $\{h_n\}$ are real sequences, $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuous, $a_n > 0$, $b_n > 0$, and $q_n > 0$ for all $n \geq n_0 \in \boldsymbol{N}_0$, $uf(u) > 0$ for $u \neq 0$, and m is a positive integer. A solution of (1) is a real sequence $\{y_n\}$ defined for all $n \geq n_0 - m + 1$ and satisfying (1) for all $n > n_0$. In what follows, we assume that equation (1) has solutions which are nontrivial and defined for all large n. A nontrivial solution $\{y_n\}$ of (1) is said to be *oscillatory* if for any $N \geq n_0$ there exists $n > N$ such that $y_{n+1}y_n \leq 0$. Otherwise, the solution is said to be *nonoscillatory*. Equation (1) is said to be *oscillatory* if every solution of (1) is oscillatory, and it is said to be *almost oscillatory* if every solution $\{y_n\}$ is either oscillatory or satisfies $\lim_{n\to\infty} \Delta^i y_n = 0$ for $i = 0, 1, 2$.

Determining oscillation criteria for third order nonlinear difference equations has not received a great deal of attention in the literature even though such equations arise in the study of economics, mathematical biology, and other areas of mathematics in which discrete models are used (see, for example, [2]). Some recent results on third order equations can be found in [1, 4–9]. In Section 2, we obtain sufficient conditions for the oscillation or almost oscillation of equation (1). A necessary and sufficient condition for almost oscillation is obtained for a special case of (1). Section 3 contains a result giving sufficient conditions for equation (1) to have a bounded nonoscillatory solution converging to a nonzero constant. Examples to illustrate the results are included.

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2. Oscillation theorems

In this section, we also assume that $\Delta a_n \geq 0$ for all $n \geq n_0$ and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty. \tag{2}$$

Our first two results are for the unforced equation, i.e., $h_n \equiv 0$ for all $n \in \boldsymbol{N}_0$.

Theorem 1. *Let $f(u) = u$ and $h_n \equiv 0$. Assume that there exist real valued functions h, $H : \boldsymbol{N}_0 \times \boldsymbol{N}_0 \to \boldsymbol{R}$ such that*

$$\begin{aligned} H(n,n) &= 0 \qquad \text{for } n \geq n_0 \geq 0, \\ H(n,s) &> 0 \qquad \text{for } n > s \geq n_0, \\ \Delta_2 H(n,s) &\leq 0 \qquad \text{for } n > s \geq n_0, \qquad \text{and} \\ -\Delta_2 H(n,s) &= h(n,s)\sqrt{H(n,s)} \qquad \text{for all } n > s \geq n_0, \end{aligned}$$

where $\Delta_2 H(n,s) = H(n,s+1) - H(n,s)$. If for every $n_1 > n_0 + m$

$$\limsup_{n\to\infty} \frac{1}{H(n,n_1)} \sum_{s=n_1}^{n-1} \left[H(n,s)q_s - \frac{a_s b_{s-m} h^2(n,s)}{4(s-m-n_0)} \right] = \infty, \tag{3}$$

and

$$\sum_{i=n}^{n+m-1} q_i \left[\sum_{j=n}^{i} \frac{1}{b_j} \left(\sum_{k=j}^{i} \frac{1}{a_k} \right) \right] > 1, \tag{4}$$

then every solution of equation (1) *is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $\{y_n\}$ is eventually positive. Then, there exists an integer $n_1 \geq n_0$ such that $y_n > 0$ and $y_{n-m} > 0$ for all $n \geq n_1$. From equation (1), we have

$$\Delta(a_n \Delta(b_n \Delta y_n)) = -q_n y_{n-m+1}$$

so that

$$\Delta(a_n \Delta(b_n \Delta y_n)) < 0 \qquad \text{for } n \geq n_1. \tag{5}$$

Thus, $\{\Delta(b_n \Delta y_n)\}$ and $\{\Delta y_n\}$ are monotone and eventually of one sign. We claim that there is an integer $n_2 \geq n_1$ such that

$$\Delta(b_n \Delta y_n) > 0 \tag{6}$$

for $n \geq n_2$. To prove this, suppose, to the contrary, that $\Delta(b_n \Delta y_n) \leq 0$. Since $q_n > 0$ and $a_n > 0$, it is clear that there is an integer $n_3 \geq n_2$ such that $a_{n_3}\Delta(b_{n_3}\Delta y_{n_3}) < 0$. Then, for $n \geq n_3$, we have

$$a_n \Delta(b_n \Delta y_n) \leq a_{n_3}\Delta(b_{n_3}\Delta y_{n_3}) < 0. \tag{7}$$

Summing inequality (7) from n_3 to $n-1$, we obtain

$$b_n \Delta y_n - b_{n_3}\Delta y_{n_3} < a_{n_3}\Delta(b_{n_3}\Delta y_{n_3}) \sum_{s=n_3}^{n-1} \frac{1}{a_s}.$$

In view of (2), we see that $b_n \Delta y_n \to -\infty$ as $n \to \infty$. Summing again, we obtain a contradiction to $y_n > 0$, and so (6) holds.

Next, we consider the following two cases. *Case* 1: $\Delta y_n \geq 0$ for $n \geq n_2$. Define

$$z_n = \frac{a_n \Delta(b_n \Delta y_n)}{y_{n-m}};$$

then $z_n > 0$ for $n \geq n_2$, and

$$\Delta z_n \leq -q_n - \frac{\Delta y_{n-m}}{y_{n-m}} z_{n+1}. \tag{9}$$

On the other hand, by (5), (6), and the fact that $\Delta a_n \geq 0$, it follows that

$$\Delta^2(b_n \Delta y_n) \leq 0. \tag{10}$$

Now (10) implies that $\Delta(b_n \Delta y_n)$ is nonincreasing, so the equality $b_n \Delta y_n = b_N \Delta y_N + \sum_{s=N}^{n-1} \Delta(b_s \Delta y_s)$, $n \geq N \geq n_2$, implies that

$$b_n \Delta y_n \geq (n-N)\Delta(b_n \Delta y_n), \tag{11}$$

and so

$$\Delta y_{n-m} \geq \frac{(n-m-N)a_n \Delta(b_n \Delta y_n)}{a_n b_{n-m}} \tag{12}$$

for $n > N + m + 1 = M$. From (9) and (12), we obtain

$$\Delta z_n \leq -q_n - \frac{(n-m-N)a_n \Delta(b_n \Delta y_n)}{y_{n-m} a_n b_{n-m}} z_{n+1}. \tag{13}$$

Also, from (5) and the fact that $\Delta y_n \geq 0$, (13) yields

$$\Delta z_n \leq -q_n - \frac{(n-m-N)}{a_n b_{n-m}} z_{n+1}^2.$$

Thus, for all $n \geq M$,

$$\sum_{n=M}^{n-1} H(n,s)q_s \leq H(n,M)z_M - \sum_{s=M}^{n-1}\left[z_{s+1}(-\Delta_2 H(n,s)) + \frac{(s-m-N)H(n,s)}{a_s b_{s-m}} z_{s+1}^2\right]$$

$$= H(n,M)z_M - \sum_{s=M}^{n-1}\left[\frac{(s-m-N)}{a_s b_{s-m}} H(n,s)z_{s+1}^2 + \sqrt{H(n,s)}h(n,s)z_{s+1}\right]$$

$$\leq H(n,M)z_M + \sum_{s=M}^{n-1} \frac{a_s b_{s-m} h^2(n,s)}{4(s-m-N)}.$$

Hence, for all $n \geq M$, we have

$$\limsup_{n\to\infty} \frac{1}{H(n,M)} \sum_{s=M}^{n-1}\left[H(n,s)q_s - \frac{a_s b_{s-m} h(n,s)^2}{4(s-m-N)}\right] \leq z_M,$$

which contradicts (3).

Case 2: $\Delta y_n < 0$ for all $n \geq n_2$. Summing equation (1) from s to n, we have

$$a_{n+1}\Delta(b_{n+1}\Delta y_{n+1}) - a_s\Delta(b_s\Delta y_s) + \sum_{i=s}^{n} q_i y_{i-m+1} = 0,$$

and so

$$-\Delta(b_n\Delta y_n) + \frac{1}{a_n}\sum_{i=n}^{\infty} q_i y_{i-m+1} \leq 0$$

by (6). Summing again from s to n and using the fact that $b_n\Delta y_n < 0$, it follows that

$$b_n\Delta y_n + \sum_{s=n}^{\infty}\left(\sum_{j=n}^{i}\frac{1}{a_j}\right) q_i y_{i-m+1} \leq 0. \tag{14}$$

A final summation of (14) yields

$$\sum_{i=n}^{\infty}\left[\sum_{j=n}^{i}\frac{1}{b_j}\left(\sum_{k=j}^{i}\frac{1}{a_k}\right)\right] q_i y_{i-m+1} \leq y_n,$$

or

$$\sum_{i=n}^{n+m-1}\left[\sum_{j=n}^{i}\frac{1}{b_j}\left(\sum_{k=j}^{i}\frac{1}{a_k}\right)\right] q_i y_{i-m+1} \leq y_n. \tag{15}$$

Since $\{y_n\}$ is decreasing, (15) yields

$$\sum_{i=n}^{n+m-1} q_i \left[\sum_{j=n}^{i} \frac{1}{b_j} \left(\sum_{k=j}^{i} \frac{1}{a_k} \right) \right] \leq 1,$$

which contradicts (4). The proof is now complete.

Example 1. The difference equation

$$(\mathrm{E}_1) \qquad \Delta^3 y_n + \frac{8n+12}{n-m+1} y_{n-m+1} = 0, \qquad n \geq m,$$

where m is a odd positive integer, satisfies all conditions of Theorem 1 for $H(n,s) = (n-s)$ and $h(n,s) = 1/\sqrt{(n-s)}$. Hence, all solutions of equation (E_1) are oscillatory. In fact $\{y_n\} = \{(-1)^n n\}$ is such a solution of (E_1).

Theorem 2. *Assume that* $h_n \equiv 0$ *and*

$$(16) \qquad f(u) - f(v) = g(u,v)(u-v) \qquad \text{and} \qquad g(u,v) \geq \mu$$

for some $\mu > 0$. *If there exists a positive sequence* $\{\rho_n\}$ *such that for every* $n_1 > n_0 + m$,

$$(17) \qquad \sum_{n=n_1}^{\infty} \left[\rho_n q_n - \frac{a_n b_{n-m} (\Delta \rho_n)^2}{4\mu \rho_n (n-m-n_0)} \right] = \infty,$$

and

$$(18) \qquad \limsup_{n\to\infty} \sum_{i=n}^{n+m-1} q_i \left[\sum_{j=n}^{i} \frac{1}{b_j} \left(\sum_{k=j}^{i} \frac{1}{a_k} \right) \right] = \infty,$$

then every solution of equation (1) *is oscillatory.*

Proof. Proceeding exactly as in Theorem 1, we see that (6) holds. Now, if $\{\Delta y_n\}$ is eventually positive, then we set

$$z_n = \frac{a_n \Delta(b_n \Delta y_n) \rho_n}{f(y_n)},$$

for $n \geq n_2$. It is easy to see that $z_n > 0$ and

$$\Delta z_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} z_{n+1} - \frac{g(y_{n-m+1}, y_{n-m}) \Delta y_{n-m} \rho_n}{f(y_{n-m}) \rho_{n+1}} z_{n+1}.$$

Using (12) and the fact that $\{a_n \Delta(b_n \Delta y_n)\}$ is nonincreasing, and $\{y_n\}$ is nondecreasing, we see that

$$\Delta z_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} z_{n+1} - \frac{\mu(n-m-N)\rho_n}{a_n b_{n-m} \rho_{n+1}} z_{n+1}^2,$$

for $n \geq N + m + 1 = M > n_2$, and completing the square, we obtain

$$\Delta z_n \leq -\rho_n q_n + \frac{a_n b_{n-m} (\Delta \rho_n)^2}{4\mu \rho_n (n-m-N)}, \qquad n \geq M.$$

Summing the last inequality from M to n and letting $n \to \infty$, we see, in view of condition (17), that $\lim_{n\to\infty} z_n = -\infty$. This contradicts the fact that $\{z_n\}$ is eventually positive.

Now assume that $\{\Delta y_n\}$ is eventually negative. By summing equation (1) three times as in the proof of Theorem 1, we see that

$$\sum_{i=n}^{\infty} \left[\sum_{j=n}^{i} \frac{1}{b_j} \left(\sum_{k=j}^{i} \frac{1}{a_k} \right) \right] q_i f(y_{i-m+1}) \leq y_n. \tag{19}$$

Since $\{y_n\}$ is decreasing and $f(u)$ is increasing, it follows from (19) that

$$\sum_{i=n}^{n+m-1} \left[\sum_{j=n}^{i} \frac{1}{b_j} \left(\sum_{j=k}^{i} \frac{1}{a_k} \right) \right] q_i \leq \frac{y_n}{f(y_n)}. \tag{20}$$

Clearly, $\lim_{n\to\infty} y_n = b \geq 0$. In view of (18), (20) implies that $b > 0$ is not possible. If $b = 0$, then

$$\lim_{n\to\infty} \frac{y_n}{f(y_n)} = \lim_{n\to\infty} \frac{1}{g(y_{n+1}, y_n)} \leq \frac{1}{\mu},$$

which of course contradicts (18). The proof for the case $y_n < 0$ for large n is similar and hence omitted. Thus, the proof is complete.

Example 2. The difference equation

$$\Delta(n\Delta(n\Delta y_n)) + 4n^3(y_{n-4}^{1/3} + y_{n-4}) = 0, \qquad n \geq 1, \tag{E$_2$}$$

satisfies all the conditions of Theorem 2 for $\rho_n \equiv 1$. Hence, all solutions of equation (E$_2$) are oscillatory.

Theorem 3. *Assume that conditions* (16) *and* (18) *hold. Suppose there exists a positive sequence* $\{\rho_n\}$ *and an oscillatory sequence* $\{\phi_n\}$ *such that*

$$\Delta(a_n \Delta(b_n \Delta \phi_n)) = h_n, \qquad \lim_{n\to\infty} \Delta^i \phi_n = 0, \qquad \text{for } i = 0, 1, 2, \tag{21}$$

and for some $\lambda \in (0,1)$ *and every* $n_1 > n_0 + m + 1$,

(22) $$\sum_{n=n_1}^{\infty}\left(\rho_n q_n - \frac{a_n b_{m-n}(\Delta\rho_n)^2}{4\mu\lambda(n-m-n_0)\rho_n}\right) = \infty.$$

Then equation (1) *is almost oscillatory.*

Proof. Suppose there is a nonoscillatory solution $\{y_n\}$ such that $\{y_n\}$ is eventually positive and $\lim_{n\to\infty} y_n = 0$. Consider the function x_n defined by

(23) $$x_n = y_n - \phi_n.$$

It is easy to see that x_n is eventually positive, for otherwise we would have $y_n < \phi_n$, and this would contradict the oscillatory nature of $\{\phi_n\}$. From equation (1), we see that

(24) $$\Delta(a_n\Delta(b_n\Delta x_n)) \leq 0.$$

Hence, $\{\Delta x_n\}$ and $\{\Delta^2 x_n)$ are monotonic and eventually of one sign. Arguing as before, it follows that there is an integer $n_1 \geq n_0$ such that

$$\Delta(b_n\Delta x_n)) > 0 \qquad \text{and} \qquad \Delta^2(b_n\Delta x_n)) \leq 0$$

for $n \geq n_1$. Now assume that $\{\Delta x_n\}$ is eventually positive. Since $\{x_n\}$ is eventually positive and increasing, and $\phi_n \to 0$ as $n \to \infty$, it follows from (23) that there exists an integer $n_2 \geq n_1$ such that

(25) $$y_{n-m+1} \geq \lambda x_{n-m+1} \qquad \text{for } n \geq n_2.$$

Therefore,

(26) $$f(y_{n-m+1}) \geq f(\lambda x_{n-m+1}).$$

Define

$$z_n = \frac{a_n\Delta(b_n\Delta x_n)}{f(\lambda x_{n-m})}\rho_n, \qquad n \geq n_2,$$

and observe that $z_n > 0$ for $n \geq n_2$ and

$$\Delta z_n \leq -q_n\rho_n\frac{f(y_{n-m+1})}{f(\lambda x_{n-m+1})} + \frac{\Delta\rho_n}{\rho_{n+1}}z_{n+1} - \frac{\lambda g(\lambda x_{n-m+1}, \lambda x_{n-m})\Delta x_{n-m}\rho_n}{f(\lambda x_{n-m})\rho_{n+1}}z_{n+1}.$$

By (26) and (16), we have

$$\Delta z_n \leq -q_n\rho_n + \frac{\Delta\rho_n}{\rho_{n+1}}z_{n+1} - \frac{\lambda\mu\Delta x_{n-m}\rho_n}{f(\lambda x_{n-m})\rho_{n+1}}z_{n+1}.$$

We now proceed as in the proof of Theorem 2 and obtain a contradiction to (22). Thus, $\{x_n\}$ must be eventually negative. In this case, $\{x_n\}$ decreases to a nonnegative constant c. Since $\lim_{n\to\infty} \phi_n = 0$, from (23) we have that $\lim_{n\to\infty} y_n = c$. Summing equation (1) three times as we did in the previous theorems, we have

$$\sum_{i=n}^{\infty}\left[\sum_{j=n}^{i}\frac{1}{b_j}\left(\sum_{k=j}^{\infty}\frac{1}{a_k}\right)\right]q_i f(y_{i-m+1}) \leq x_n.$$

Hence, $\liminf_{n\to\infty} y_n = 0$, and since $\{y_n\}$ is monotonic, we have $\lim_{n\to\infty} y_n = 0$. Thus, $c = 0$, and by (21) and (23), we have $\lim_{n\to\infty} \Delta^i y_n = 0$ for $i = 0, 1, 2$. This completes the proof of the theorem.

Example 3. The difference equation

$$(\mathrm{E}_3) \qquad \Delta[(n+1)\Delta((n+1)\Delta y_n)] + n^4(y_{n-3}^3 + y_{n-3}) = \frac{(-1)^{n+1}[8n^3 + 24n^2 + 18n + 1]}{n(n+1)}, \qquad n \geq 1,$$

satisfies all conditions of Theorem 3 for $\rho_n \equiv 1$ and $\{\phi_n\} = \{(-1)^n/n\}$. Hence, all solutions of equation (E_3) are almost oscillatory.

Remark. If $f(u) = u$, then condition (18) of Theorems 2 and 3 can be replaced by condition (4) of Theorem 1.

Next, we study the almost oscillation of equation (1) when $a_n \equiv 1$.

Theorem 4. *Assume that* (16) *holds and that there exists a positive sequence* $\{\rho_n\}$ *and an oscillatory sequence* $\{\phi_n\}$ *such that*

$$\Delta^2(b_n\Delta\phi_n)) = h_n, \qquad \lim_{n\to\infty} \Delta^i\phi_n = 0, \qquad \textit{for } i = 0, 1, 2,$$

$\Delta\rho_n \leq 0$, *and* $\Delta^2\rho_n \geq 0$ *for* $n \geq n_0$. *If*

$$(27) \qquad \sum_{n=n_0}^{\infty} q_n\rho_n = \infty$$

and

$$(28) \qquad \sum_{n=n_0}^{\infty}\frac{1}{b_n\rho_n}\sum_{s=n}^{\infty}(s-n+1)q_s\rho_{s+2} = \infty,$$

then equation (1) *is almost oscillatory.*

Proof. Similar to the proof of Theorem 3, we obtain

$$\Delta z_n \leq -q_n\rho_n + \frac{\Delta\rho_n}{\rho_{n+1}}z_{n+1} - \frac{g(\lambda x_{n-m+1}, \lambda x_{n-m})\Delta x_{n-m}\rho_n}{f(\lambda x_{n-m})\rho_{n+1}}z_{n+1}$$

for $n \geq N$ for some $N \geq n_0$. The hypotheses on $\{\rho_n\}$ then yield

$$\Delta z_n \leq -q_n\rho_n.$$

Summing the last inequality from N to n and letting $n \to \infty$, we obtain a contradiction to (27). Thus, $\{\Delta x_n\}$ must be eventually negative, and so $\{x_n\}$ decreases to $c \geq 0$. Since $\lim_{n\to\infty} \phi_n = 0$, (23) implies that $\lim_{n\to\infty} y_n = c$. We will prove that $c = 0$, so suppose $c > 0$. Then there is an integer $N_1 \geq N > 0$ such that

$$y_{n-m+1} \geq \frac{c}{2}$$

for all $n \geq N_1$. Let $w_n = b_n\rho_n\Delta x_n$. Then

$$\Delta^2 w_n = -q_n f(y_{n-m+1})\rho_{n+2} + 2\Delta\rho_{n+1}\Delta(b_n\Delta x_n) + (b_n\Delta x_n)\Delta^2\rho_n.$$

Since $\Delta\rho_n < 0$ and $\Delta^2\rho_n > 0$, we have

$$\Delta^2 w_n + q_n\rho_{n+2}f(c/2) \leq 0$$

for $n \geq N_1$. Summing the last inequality from n to j and using the fact that $\Delta w_j > 0$, it follows that

$$-\Delta w_n + f\left(\frac{c}{2}\right)\sum_{s=n}^{j} q_s\rho_{s+2} \leq 0.$$

Letting $j \to \infty$, we have

$$-\Delta w_n + f\left(\frac{c}{2}\right)\sum_{s=n}^{\infty} q_s\rho_{s+2} \leq 0.$$

Summing again and now using the fact that $w_j < 0$, we find that

$$\Delta x_n \leq -\frac{f(c/2)}{b_n\rho_n}\sum_{s=n}^{\infty}(s-n+1)q_s\rho_{s+2}.$$

Summing the last inequality from N_1 to $n-1$, we obtain

$$x_n \leq x_{N_1}\sum_{s=N_1}^{n-1}\frac{1}{b_s\rho_s}\sum_{t=s}^{\infty}(t-s+1)q_t\rho_{t+2}.$$

By (28), we have $\lim_{n\to\infty} x_n = -\infty$, which is a contradiction and completes the proof of the theorem.

Example 4. Consider the difference equation

$$(\mathrm{E}_4) \qquad \Delta^2((n+1)\Delta y_n) + \frac{4n^4+16n^3+18n^2+4n-1}{n(n+1)(n+2)}(y_{n-2}^{1/3}+y_{n-2})$$
$$= (-1)^{n+1}\left(\frac{8n^3+28n^2+24n+2}{n(n+1)(n+2)}\right), \qquad n \geq 5.$$

The above equation satisfies all the conditions of Theorem 4 for $\rho_n \equiv 1$ and $\{\phi_n\} = \{(-1)^n/n\}$. Therefore, all solutions are almost oscillatory, and $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of equation (E_4).

Example 5. Consider the equation

$$(\mathrm{E}_5) \quad \Delta^2((n+1)\Delta y_n) + \left(\frac{2^{n-6}}{4^n+1}\right)((n-1)2^{n+3} - (-1)^n(125n+175))(y_n^3+y_n)$$
$$= (-1)^{n+1}\left(\frac{125n+175}{4^{n+3}}\right), \qquad n \geq 6.$$

All hypotheses of Theorem 4 are satisfied with $\rho_n \equiv 1$ and $\{\phi_n\} = \{(-1)^n/4^n\}$. Thus, all solutions of (E_5) are almost oscillatory; $\{y_n\} = \{1/2^n\} \to 0$ is a solution of equation (E_5).

Theorem 5. *Let* $a_n = b_n = 1$ *and suppose that* (21) *holds. If*

$$(29) \qquad \sum_{n=n_0}^{\infty} n^2 q_n = \infty,$$

then every bounded solution $\{y_n\}$ *of* (1) *is almost oscillatory.*

Proof. Letting $x_n = y_n - \phi_n$ and proceeding as in the proof of Theorem 3, we see that $\{x_n\}$ is eventually positive, $\{\Delta x_n\}$ and $\{\Delta^2 x_n\}$ are monotonic and eventually of one sign, and $\{\Delta^2 x_n\}$ is nondecreasing.

Suppose that $\{\Delta^2 x_n\}$ is eventually negative. Then, since $\{\Delta x_n\}$ is decreasing and concave down, it must eventually become negative. By the same reasoning, $\{x_n\}$ will eventually be negative, and this contradiction proves that $\Delta^2 x_n \geq 0$ eventually.

Now if $\{\Delta x_n\}$ is eventually positive, then $x_n > 0$ and $\Delta^2 x_n \geq 0$ imply $\{x_n\}$ becomes unbounded, which is a contradiction. Thus, $\{\Delta x_n\}$ is eventually negative, and so there exists a nonnegative number c such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = c$. To complete the proof, it suffices to show that $c = 0$, so assume $c > 0$. Summing equation (1) and using the fact that $\Delta^2 x_n > 0$, we have

$$-\Delta^2 x_n + \sum_{s=n}^{\infty} q_s f(y_{s-m+1}) \leq 0.$$

Repeating the above procedure, we have

$$\Delta x_n + \sum_{s=n}^{\infty}(s-n+1)q_s f(y_{s-m+1}) \leq 0,$$

and summing one last time from N to ∞, we obtain

$$c - x_N + \sum_{s=N}^{\infty}\frac{(s-N+1)(s-N+2)}{2}q_s f(y_{s-m+1}) \leq 0. \tag{30}$$

By (29), it follows from (30) that

$$\liminf_{n\to\infty} f(y_{n-m+1}) = \lim_{n\to\infty} x_n = 0.$$

This contradiction completes the proof of the theorem.

Example 6. The difference equation

$$\Delta^3 y_n + \frac{2^{2n-3}}{27000}(30 - 27(-1)^n)y_n^3 = \frac{27(-1)^{n+1}}{2^{n+3}}, \qquad n \geq 2, \tag{E$_6$}$$

satisfies all conditions of Theorem 5 with $\{\phi_n\} = \{(-1)^n/2^n\}$. Hence, every bounded solution of equation (E_6) is almost oscillatory, and $\{y_n\} = \{30/2^n\}$ is one such solution.

In our next theorem, we do not require that q_n be positive.

Theorem 6. *Let $a_n = b_n \equiv 1$ and f be nondecreasing. If*

$$\sum_{n=n_0}^{\infty} n^2|q_n| < \infty$$

and

$$\sum_{n=n_0}^{\infty} n^2|h_n| < \infty, \tag{31}$$

then equation (1) *has a nonoscillatory solution that approaches a nonzero real number as $n \to \infty$.*

Proof. Let $c > 0$ be given and choose N so that

$$\sum_{n=N}^{\infty} n^2|q_n| < \frac{c}{2f(2c)} \tag{32}$$

and

$$\sum_{n=N}^{\infty} n^2|h_n| < \frac{c}{2}. \tag{33}$$

Let $\mathscr{B}_N$ be the Banach space of all real sequences $Y = \{y_n\}, n \geq N$, with norm

$$\|Y\| = \sup_{n \geq N} |y_n|.$$

Let $\mathscr{S} = \{Y \in \mathscr{B}_N : c \leq y_n \leq 2c, n \geq N\}$ and define $T : \mathscr{S} \to \mathscr{B}_N$ by

$$(TY)_n = \frac{3c}{2} + \frac{1}{2}\sum_{s=n}^{\infty}(s - n + 1)(s - n + 2)(q_s f(y_{s-m+1}) - h_s), \qquad n \geq N.$$

Clearly, $\mathscr{S}$ is a bounded, closed, and convex subset of $\mathscr{B}_N$.

First, we will show that T maps $\mathscr{S}$ into itself. For any $Y \in \mathscr{S}$, we have

$$\left|(TY)_n - \frac{3c}{2}\right| \leq \frac{1}{2}\sum_{s=n}^{\infty} s^2(|q_s| f(2c) + |h_s|) \leq \frac{c}{2}$$

for $n \geq N$. Thus, $T\mathscr{S} \subset \mathscr{S}$.

Next, we let $X = \{x_n\} \in \mathscr{S}$ and for each $i = 1, 2, \ldots$ let $Y^i = \{y_n^i\}$ be a sequence in $\mathscr{S}$ such that $\lim_{i\to\infty} \|Y^i - X\| = 0$. Then a straight forward argument using the continuity of f shows that $\lim_{i\to\infty} \|(TY^i)_n - (TX)_n\| = 0$, and so T is continuous.

Finally, in order to apply Schauder's fixed point theorem, we need to show that $T\mathscr{S}$ is relatively compact. In view of a recent result of Cheng and Patula [3], it suffices to show that $T\mathscr{S}$ is uniformly Cauchy. To this end, let $Y = \{y_n\} \in \mathscr{S}$ and observe that for any $k > n > N$, we have

$$|Ty_n - Ty_k| \leq \sum_{s=n}^{\infty} s^2(|q_s| f(2c) + |h_s|).$$

From the hypotheses, it is clear that for a given $\varepsilon > 0$ there exists an integer N_1 such that for all $k > n \geq N_1$, we have $|Ty_n - Ty_k| < \varepsilon$. Thus, $T\mathscr{S}$ is uniformly Cauchy, and so $T\mathscr{S}$ is relatively compact. Therefore, by Schauder's fixed point theorem, there is a fixed point $Y \in \mathscr{S}$. It is clear that $Y = \{y_n\}$ is a nonoscillatory solution of (1) for $n \geq N$ and has the required properties.

Example 7. Consider the difference equation

$$(E_7) \qquad \Delta^3 y_n + \frac{2^{-n}[(1/8) - (-1)^n 2^{-n}]}{(1 + 2^{-n})} y_n = (-1)^{n+1} 2^{-2n}, \qquad n \geq 3.$$

With $\{\phi_n\} = \{(4/5)^3(-1)^n/4^n\}$, all conditions of Theorem 6 are satisfied. Hence, equation (E_7) has a nonoscillatory solution that approaches a non-zero real number. In fact, $\{y_n\} = \{1 + 2^{-n}\}$ is such a solution of equation (E_7).

By combining Theorem 5 and 6, we have the following necessary and sufficient condition for the almost oscillation of equation (1).

Theorem 7. *Let* $a_n = b_n = 1$, f *be nondecreasing, and conditions* (21) *and* (31) *hold. Then every bounded solution of equation* (1) *is almost oscillatory if and only if* (29) *holds.*

3. Asymptotic behavior

In this section, we obtain a sufficient condition for the asymptotic behavior of solutions of equation (1). We do not require $q_n > 0$ here. Let A_n, B_n, and C_n be defined by

$$A_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}, \qquad B_n = \sum_{s=n_0}^{n-1} \frac{1}{b_s} \qquad \text{and} \qquad C_n = \sum_{s=n_0}^{n-1} \frac{A_s}{b_s}.$$

Theorem 8. *Let* $f(u)$ *be nondecreasing and let* $d > 0$ *be a constant such that* $a_n \geq d$ *for all* $n \geq n_0$. *Suppose that*

$$\sum_{n=n_0}^{\infty} [C_{n+1} + A_{n+1}B_{n+1}]|h_n| < \infty \tag{34}$$

and

$$\sum_{n=n_0}^{\infty} [C_{n+1} + A_{n+1}B_{n+1}]|q_n| < \infty. \tag{35}$$

Then equation (1) *has a bounded nonoscillatory solution that approaches a nonzero limit.*

Proof. Let $c > 0$ and let N be so large that

$$\sum_{n=N}^{\infty} [C_{n+1} + A_{n+1}B_{n+1}]|h_n| < \frac{c}{4} \tag{36}$$

and

$$\sum_{n=N}^{\infty} [C_{n+1} + A_{n+1}B_{n+1}]|q_n| < \frac{c}{4f(2c)}. \tag{37}$$

Let the Banach space $\mathscr{B}_N$ and the set $\mathscr{S} \subseteq \mathscr{B}_N$ be the same as in Theorem 6 and define the operator $T : \mathscr{S} \to \mathscr{B}_N$ by

$$(Ty)_n = \frac{3c}{2} - \sum_{s=n}^{\infty} K(s,n)(q_s f(y_{s-m+1}) - h_s), \qquad n \geq N,$$

where $K(s,n) = C_{s+1} - C_n + A_{s+1}B_n - A_{s+1}B_{s+1}$. Similar to the proof of Theorem 6, we can show that the mapping T satisfies the hypotheses of Schauder's fixed point theorem. Hence, T has a fixed point $Y \in \mathscr{S}$, and it is clear that $Y = \{y_n\}$ is a nonoscillatory solution of equation (1) for $n \geq N$ and has the desired properties.

It should be pointed out that Theorem 6 is actually a special case of the above result. We conclude this paper with a simple example of Theorem 8.

Example 8. Consider the equation

$$(\mathrm{E}_8) \qquad \Delta^2(n^3 \Delta y_n) + (-1)^n 3^{-n} y_{n-m}^{\gamma} = (-1)^{n+1} 2^{-n}, \qquad n \geq 1,$$

where γ is the ratio of odd positive integers and m is a positive integer. All conditions of Theorem 8 are satisfied, so equation (E_8) has a bounded nonoscillatory solution that approaches a non-zero limit.

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