Large Time Behavior of Solutions for Derivative Cubic Nonlinear Schrödinger Equations without a Self-Conjugate Property

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§1. Introduction

In this paper we study the Cauchy problem for the derivative cubic nonlinear Schrödinger equation of the following form

\[\begin{align*}
  iu_t + u_{xx} &= \mathcal{N}(u), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
  u(0, x) &= u_0, \quad x \in \mathbb{R},
\end{align*}\]  

(1.1)

where \( \mathcal{N}(u) = ia(|u|^2 u)_x + \lambda_1 (u^3)_x + \lambda_2 (\overline{u}^2 u)_x + \lambda_3 (\overline{u}^3)_x, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}, \quad a \in \mathbb{R}. \) The linear part of equation (1.1) consists of the linear Schrödinger operator, while the nonlinearity involves all full derivatives of unknown function of the cubic order. Such kinds of equations are of highest interest in many areas of Physics. The difficulty in the study of the large time asymptotic behavior of solutions to the Cauchy problem (1.1) is that the cubic nonlinear term of (1.1) is critical for large time values, and it is already known that the usual scattering states do not exist for derivative nonlinear Schrödinger equation (1.1) in the case \( \lambda_j = 0, \ j = 1, 2, 3, \ a \neq 0. \) Also note that the structure of the nonlinearity does not possess the following self-conjugate property: \( \mathcal{N}(e^{i\theta}u) = e^{i\theta}\mathcal{N}(u) \) for all \( \theta \in \mathbb{R}, \) so we can not apply directly the usual operators \( \mathcal{J} = x + 2it\partial_x \) for the study of the large time behavior of solutions. There are some works (see, [4, 5, 9, 10, 12, 13, 16]) concerning with the large time asymptotics of solutions to the derivative nonlinear Schrödinger equations with cubic nonlinearities which have the self-conjugate property. Recent developments in this direction can be seen in [9]. In the present paper we are interested in the asymptotic behavior of solutions to the nonlinear Schrödinger equations with nonlinearities which do not have a self-conjugate property. We prove the global existence of solutions to the Cauchy problem (1.1) in the weighted Sobolev spaces for small initial data as well as the existence of the usual and modified scattering states (see Theorem 1.1 below). Furthermore we obtain the large time asymptotics of solutions (involving the sharp \( L^\infty \) time decay estimates). To treat the critical
cubic nonlinearity of (1.1) we use the techniques developed in our previous works [6], [7], where we introduced an appropriate representation of the solution and instead of the operator $\mathcal{I}$ we used the operator $\mathcal{I} = x + 2it\partial_x \partial_x^{-1}$, which is related to the dilation operator.

We now give some notation and function spaces. Let $\mathcal{F}\phi$ or $\hat{\phi}$ denote the Fourier transform of $\phi$ defined by $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-it\xi} \phi(x) \, dx$. The inverse Fourier transform $\mathcal{F}^{-1}\phi$ or $\check{\phi}$ is given by $\check{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int e^{it\xi} \phi(\xi) \, d\xi$. The free Schrödinger evolution group $\mathcal{U}(t)$ is written as $\mathcal{U}(t)\phi = \mathcal{F}^{-1}(\mathcal{F} \phi) = e^{-it\mathcal{F}} \phi$ and also can be represented in the following form $\mathcal{U}(t) = M(t)\mathcal{K}(t)\mathcal{F} M(t)$, where $M(t) = \exp(i\frac{x^2}{4t})$, the dilation operator is $(\mathcal{K}(t)\phi)(x) = \frac{1}{\sqrt{-2it}} \phi \left( \frac{x}{2t} \right)$. The inverse free Schrödinger evolution group is $\mathcal{U}(-t) = -M(-t)i\mathcal{F}^{-1}\mathcal{K} \left( \frac{1}{2t} \right) M(-t)$, where $\mathcal{K}^{-1}(t) = -i\mathcal{K} \left( \frac{1}{2t} \right)$ is the inverse dilation operator. Using the above identities we easily see that $\mathcal{I}(t) = x + 2it\partial_x = \mathcal{U}(t)x\mathcal{U}(-t) = M(t)(2it\partial_x) \cdot M(-t)$. It seems to be difficult to apply the operator $\mathcal{I}$ directly to equation (1.1) since the nonlinearity does not have a special complex conjugate structure as stated above. So instead of the operator $\mathcal{I}$ we introduce the operator $\mathcal{J} = x + 2it\partial_x \partial_x^{-1}$, where $\partial_x^{-1} = \int_x^\infty \, dx$. Note that $\mathcal{J}\partial_x$ is a differential operator of the first order. The operator $\mathcal{J}$ is related with the operator $\mathcal{I}$, since we have $\mathcal{J} = \mathcal{J} - 2it\partial_x \partial_x^{-1}\partial_t$. We also widely use the following commutation relations $[\mathcal{C}, \mathcal{J}] = 0, \ [\mathcal{C}, \mathcal{J}] = 2\partial_x^{-1}\partial_t, \ [\mathcal{J}, \partial_x] = [\mathcal{J}, \partial_x] = -1, \text{ where } \mathcal{C} = i\partial_t + \partial_x^2, \ \partial_t = \frac{\partial}{\partial t}$ and $\partial_x = \frac{\partial}{\partial x}$.

We introduce some function spaces. The Lebesgue space is $L^p = \{ \phi \in \mathcal{F}' \| \phi \|_p < \infty \}$, where $\| \phi \|_p = (\int |\phi(x)|^p \, dx)^{1/p}$ if $1 \leq p < \infty$ and $\| \phi \|_\infty = \text{ess.sup}\{ |\phi(x)| ; x \in \mathcal{R} \}$ if $p = \infty$. For simplicity we let $\| \phi \| = \| \phi \|_2$. Weighted Sobolev space is $H^{m,s}_p = \{ \phi \in \mathcal{F}' : \| \phi \|_{m,s,p} = \| (1 + x^2)^{m/2} (1 - \partial_x^2)^{m/2} \phi \|_p < \infty \}$, $m,s \in \mathcal{R}, \ 1 \leq p \leq \infty$; we denote also $H^{m,s}_2 = H^{m,s}_2$ and $\| \phi \|_{m,s} = \| \phi \|_{m,s,2}$. Let $C(I;B)$ be the space of continuous functions from a time interval $I$ to a Banach space $B$. Different positive constants we denote everywhere by the same letter $C$.

Our main results are the followings.

**Theorem 1.1.** Let the initial data $u_0 \in H^{2,0} \cap H^{1,1}$ and the norm $\varepsilon = \| u_0 \|_{2,0} + \| u_0 \|_{1,1}$ be sufficiently small. Then there exists a unique global solution of the Cauchy problem (1.1) such that $u \in C(R;H^{1,0}) \cap L^\infty_{loc}(R;H^{2,0})$, $\mathcal{J}u \in L^\infty_{loc}$. 
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(R; H^{1,0}), and the following time decay estimate

\begin{equation}
\|u(t)\|_{1,0,\infty} \leq C|t|^{-1/2}
\end{equation}

is valid for all t. Moreover there exist unique functions \( u^+_j \in L^2 \cap L^\infty \), and the real valued functions \( g^\pm \in L^\infty \) such that

\begin{equation}
\|\mathcal{U}(t)\partial_x^j u(t) - \mathcal{F}^{-1}(u^+_j \pm \exp(\pm ig^\pm \log |t|))\| \leq C|t|^{-\alpha},
\end{equation}

as \( t \to \pm \infty \), where \( j = 0, 1, 0 < \alpha < \frac{1}{4} \) and \( g^\pm = 0 \) when \( a = 0 \). The following asymptotic formula

\begin{equation}
\partial_x^j u(t,x) = \frac{1}{\sqrt{t}} u^+_j \left( \frac{x}{2t} \right) \exp\left( \frac{ix^2}{4t} \pm ig^\pm \left( \frac{x}{2t} \right) \log |t| \right) + O(|t|^{-1/2-\varepsilon}),
\end{equation}

is true as \( t \to \pm \infty \) uniformly in \( x \in \mathbb{R} \), where \( 0 < \alpha < \frac{1}{4} \).

Next we present the nonexistence of the usual scattering states in the case \( a \neq 0 \).

**Theorem 1.2.** In addition to the conditions of Theorem 1.1 we assume that \( a \neq 0 \) and there exists a final state \( u^+ \in H^{1,\delta}, \delta > \frac{1}{2} \) such that \( \|u(t) - \mathcal{U}(t)u^+\|_{1,0} \to 0 \) as \( t \to \infty \). Then the final state \( u^+ \) is identically zero. Furthermore if the solution satisfies \( L^2 \) conservation law, then the solution \( u \) is also identically zero.

**Remark 1.1.** In the case \( a = 0 \) the usual scattering states exist and the asymptotic formula (1.4) has a quasilinear character.

We organize our paper as follows. In Section 2 we describe a smoothing effect of the linear Schrödinger equation. Section 3 is devoted to some preliminary estimates of solutions to the Cauchy problem (1.1). The local existence of solutions for the Cauchy problem (1.1) in a space \( X^{m,s} = \{ \phi \in C([0,T]; H^{m-1,0}) \cap L^\infty(0,T; H^{m,0}); \mathcal{J}^s \phi \in L^\infty(0,T; H^{m-s,0}) \} \), where \( m \geq 2 \), \( 0 \leq s \leq m \), is stated in Theorem 3.1. Then in our key Lemma 3.1 we prove the optimal time decay estimate of global solutions to the Cauchy problem (1.1) in the uniform norm \( \sup_{t \geq 0} \sqrt{1+t} \| u(t) \|_{1,0,\infty} \leq C \), while the norm of solutions \( \| u(t) \|_Y = \| u(t) \|_{2,0} + \| \mathcal{J} u(t) \|_{1,0} \) can slightly grow with respect to time: \( \sup_{t \geq 0} (1+t)^{-\gamma} \| u(t) \|_Y \leq C \), where \( \gamma \in (0, \frac{1}{12}) \). Section 4 is devoted to the proof of Theorems 1.1–1.2. In what follows we consider the case of the positive time only since the negative time can be treated in the same manner.
§ 2. Linear smoothing effect

In this section we the Cauchy problem for the linear Schrödinger equations

\begin{equation}
\begin{cases}
  iu_t + u_{xx} = f, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\end{equation}

where the function \( f(t, x) \) is a force. We give a smoothing effect for solutions of (2.1) which is considered as a simple and explicit modification obtained by Doi [2]. We also present two lemmas which are needed to obtain the results.

The Hilbert transformation with respect to the variable \( x \) is defined as follows

\[
\mathcal{H} \phi(x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\phi(z)}{x - z} dz = -i \mathcal{F}^{-1} \frac{\xi}{|\xi|} \mathcal{F} \phi,
\]

where PV means the principal value of the singular integral. We widely use the fact that the Hilbert transformation \( \mathcal{H} \) is a bounded operator from \( \mathbb{L}^2 \) to \( \mathbb{L}^2 \). The fractional derivative \( |\partial|^\alpha \), \( \alpha \in (0, 1) \) is equal to

\[
|\partial|^\alpha \phi = \mathcal{F}^{-1} |\xi|^\alpha \mathcal{F} \phi = C \int_{\mathbb{R}} (\phi(x + z) - \phi(x)) \frac{dz}{|z|^{1+\alpha}}
\]

and similarly we have

\[
|\partial|^\alpha \mathcal{H} \phi = -i \mathcal{F}^{-1} \text{sign} \xi |\xi|^\alpha \mathcal{F} \phi = C \int_{\mathbb{R}} (\phi(x + z) - \phi(x)) \frac{dz}{z|z|^\alpha},
\]

with some constants \( C \) (see [15] for constants \( C \)). The next lemma show that commutators \( [|\partial|^\alpha, \phi] \) and \( [|\partial|^\alpha \mathcal{H}, \phi] \), are continuous operators from \( \mathbb{L}^2 \) to \( \mathbb{L}^2 \).

**Lemma 2.1.** The following inequalities

\[
||[|\partial_x|^\alpha, \phi]|\psi|| \leq C \||\phi||_{1,0,\infty} \|\psi \| \quad \text{and} \quad \||[|\partial_x|^\alpha \mathcal{H}, \phi]|\psi|| \leq C \||\phi||_{1,0,\infty} \|\psi \|
\]

are valid, provided that the right hand sides are bounded.

We define a smoothing operator \( \mathcal{S}(\varphi) = \cosh(\varphi) + i \sinh(\varphi) \mathcal{H} \), where the real-valued function \( \varphi(t, x) \in \mathbb{L}^\infty(0, T; \mathbb{H}^{2,0}_\infty(\mathbb{R})) \cap \mathbb{C}^1([0, T]; \mathbb{L}^\infty(\mathbb{R})) \) and is positive. From its definition we easily see that the operator \( \mathcal{S} \) acts continuously from \( \mathbb{L}^2 \) to \( \mathbb{L}^2 \) with the following estimate \( \|\mathcal{S}(\varphi)\psi\| \leq 2 \exp(\|\varphi\|_{\infty}) \|\psi\| \). Since \( \|\tanh(\varphi)\psi\| \leq \tanh(\|\varphi\|_{\infty}) \|\psi\| < \|\psi\| \) the inverse operator \( \mathcal{S}^{-1}(\varphi) = (1 + i \tanh(\varphi) \mathcal{H})^{-1} \frac{1}{\cosh(\varphi)} \) also exists and is continuous

\begin{equation}
\|\mathcal{S}^{-1}(\varphi)\psi\| \leq (1 - \tanh(\|\varphi\|_{\infty}))^{-1} \|\psi\| \leq \exp(\|\varphi\|_{\infty}) \|\psi\|.
\end{equation}
The operator $\mathcal{S}$ helps us to obtain a smoothing property of the Schrödinger—type equation (2.1) by virtue of the usual energy estimates. In the next lemma we present an energy estimate, involving the operator $\mathcal{S}$, in which we have an additional positive term giving us the norm of the half derivative of the unknown function $u$. We also assume that $\varphi(x)$ is written as $\varphi(x) = \delta_x^{-1}(\omega^2)$, so that $\omega(x) = \sqrt{(\partial_x \varphi)}$.

**Lemma 2.2.** The following inequality

$$\frac{d}{dt} ||\mathcal{S}u||^2 + ||\omega \mathcal{S} \sqrt{\partial_x |u|}||^2 \leq 2 |\text{Im}(\mathcal{S}u, \mathcal{S}f)|$$

$$+ C ||u||^2 e^{2||\omega||^2} (||\omega||^4 + ||\omega||_{1,0,\infty}^2 ||\omega||_{\infty} + ||\varphi||_{\infty})$$

is valid for the solution $u$ of the Cauchy problem (2.1).

In the next lemma we give the estimate for the nonlinearity.

**Lemma 2.3.** We have the following estimates

$$||\mathcal{S}u, \mathcal{S} \varphi \partial_x v)|| \leq || \varphi ||_{\mathcal{S} \sqrt{\partial_x |u|}}^2 + || \varphi ||_{\mathcal{S} \sqrt{\partial_x |v|}}^2$$

$$+ C ||u|| ||v|| \exp(6||\varphi||_{\infty})(||\varphi||_{1,0,\infty}^2 + ||\varphi||_{1,0,\infty}^2)(1 + ||\varphi||_{1,0,\infty})^2,$$

and

$$||\mathcal{S}u, \mathcal{S} \varphi \partial_x v)|| \leq || \varphi ||_{\mathcal{S} \sqrt{\partial_x |u|}}^2 + \exp(2||\varphi||_{\infty}) || \varphi ||_{\mathcal{S} \sqrt{\partial_x |v|}}^2$$

$$+ C ||u|| ||v|| \exp(6||\varphi||_{\infty})(||\varphi||_{1,0,\infty}^2 + ||\varphi||_{1,0,\infty}^2)(1 + ||\varphi||_{1,0,\infty})^2,$$

provided that the right hand sides are bounded.

For the proofs of Lemmas 2.1–2.3, see [8].

§ 3. Preliminary a-priori estimates

By virtue of the method of papers [1], [11] and [14] (see also the proof of a-priori estimates in the norm $Y$ below in Lemma 3.1) we easily obtain the existence of local solutions in the functional spaces $X^{m,s}$, with any integers $m \geq 2$, and $0 \leq s \leq m$. Below we will use this result with $m = 2$ and $s = 1$.

**Theorem 3.1.** Let the initial data $u_0 \in H^{m,0} \cap H^{m-s,s}$ with some $m \geq 2$, $0 \leq s \leq m$, $m, s \in \mathbb{N}$. Then for some time $T > 0$ there exists a unique solution $u \in X^{m,s}$ of the Cauchy problem (1.1). If we assume in addition that the norm of the initial data $|u_0|_{m,0} + |u_0|_{m-s,s} = \epsilon$ is sufficiently small, then there exists a unique solution $u \in X^{m,s}$ of (1.1) on a finite time interval $[0, T]$ with $T > 1/\epsilon$, such that the following estimate $\sup_{t \in [0,T]} (||u||_{m,0} + \|\mathcal{S}^2 u\|_{m-s,0}) < 2\epsilon$ is valid.
In the next lemma we obtain the optimal time decay estimate \( \|u(t)\|_{1,0,\infty} \leq C(1 + t)^{-1/2} \) of global solutions to the Cauchy problem (1.1) along with a-priori estimates of solutions in the norm \( Y \).

**Lemma 3.1.** Let the initial data \( u_0 \in H^{2,0} \cap H^{1,1} \) and the norm \( \|u_0\|_{2,0} + \|u_0\|_{1,1} = \epsilon \) be sufficiently small. Then there exists a unique global solution of the Cauchy problem (1.1) such that \( u \in C(R; H^{1,0}) \cap L^\infty_{loc}(R; H^{2,0}) \), and \( \mathcal{J}u \in L^\infty_{loc}(R; H^{1,0}) \). Moreover the following estimate

\[
(3.1) \quad (1 + t)^{-\gamma} \|u(t)\|_Y < 3\epsilon \quad \text{and} \quad \|u\| + \sqrt{1 + t} \|u(t)\|_{1,0,\infty} < \sqrt{\epsilon}
\]
is valid for all \( t \geq 0 \), where \( \gamma \in (0, \frac{1}{12}) \).

**Proof.** Applying the result of Theorem 3.1 and using a standard continuation argument we can find a maximal time \( T > 0 \) such that the following inequality

\[
(3.2) \quad (1 + t)^{-\gamma} \|u(t)\|_Y \leq 6\epsilon \quad \text{and} \quad \|u\| + \sqrt{1 + t} \|u(t)\|_{1,0,\infty} \leq 2\sqrt{\epsilon}
\]
is true for all \( t \in [0, T] \). If we prove (3.1) on the whole time interval \([0, T]\), then by the contradiction argument we obtain the desired result of the lemma. Note that in view of the relation \( \mathcal{J} = \mathcal{J} - 2i\partial^{-1}_x \mathcal{J} \) we get also estimate

\[
\|\mathcal{J}u\|_{1,0} \leq \|\mathcal{J}u\|_{1,0} + 2t\|\partial^{-1}_x \mathcal{N}(u)\|_{1,0} \leq \|\mathcal{J}u\|_{1,0} + C\|u\|_{1,0}^2 \|u\|_{1,0} \leq C\epsilon(1 + t)^{\gamma}.
\]

By the usual energy method (i.e., multiplying (1.1) by \( \overline{u} \), integrating over \( R \) and taking the imaginary part of the result) we get

\[
\frac{d}{dt} \|u(t)\|^2 \leq C \|u\|_{1,0,\infty}^2 \|u\|^2,
\]
whence in view of the estimate (3.2) and the Gronwall inequality we obtain the following rough estimate \( \|u(t)\| \leq 2\epsilon(1 + t)^{\gamma} \). Since the nonlinearity has a form of the full derivative, the operator \( \mathcal{J} \) acts on the nonlinear term \( \mathcal{N} \) as a differential operator. Applying the operator \( \mathcal{J} \) to both sides of equation (1.1) we find

\[
(3.3) \quad \mathcal{J} \mathcal{J}u = 2\partial^{-1}_x \mathcal{N}(u) + \mathcal{J} \mathcal{N}(u) = 2ia|u|^2u + 2\lambda_1 u^3 + 2\lambda_2 \overline{u}^2u + 2\lambda_3 u^3
\]

\[
+ ia(2|u|^2 \mathcal{J}u_x + u^2 \overline{\mathcal{J}u_x}) + 3\lambda_1 u^2 \mathcal{J}u_x + \lambda_2(\overline{u}^2 \mathcal{J}u_x + 2|u|^2 \overline{\mathcal{J}u_x})
\]

\[
+ 3\lambda_3 \overline{u}^2 \overline{\mathcal{J}u_x},
\]
whence the energy method with the estimate (3.2) yield

\[
(3.4) \quad \frac{d}{dt} \|\mathcal{J}u(t)\|^2 \leq C \|u\|_{\infty}^2 (\|u\|^2 + \|\mathcal{J}u_x\|^2) \leq C\epsilon^3(1 + t)^{2\gamma-1}.
\]
Integration of (3.4) with respect to time yields \( \| \mathcal{U} u \| \leq 2c(1 + t)^{7} \). Differentiating equation (1.1) two times with respect to \( x \) and similarly differentiating equation (3.3) we get

\[
\mathcal{C} u_{xx} = \mathcal{N}_{u_{x}} u_{xxx} + \mathcal{N}_{u_{x}} \tilde{u}_{xxx} + R_{1},
\]

and

\[
\mathcal{C}(\mathcal{U} u)_{x} = \mathcal{N}_{u_{x}} (\mathcal{U} u)_{xx} + \mathcal{N}_{u_{x}} (\mathcal{U} \tilde{u})_{xx} + R_{2},
\]

where in view of (3.2) the remainder terms \( R_{1} \) and \( R_{2} \) have the estimates \( \| R_{1} \| \leq C\| u \|_{1,0,\infty}^{2} \| u \|_{2,0} \leq Cc^{2}(1 + t)^{-1} \) and \( \| R_{2} \| \leq C\| u \|_{1,0,\infty}^{2} (\| \mathcal{U} u \|_{1,0} + \| u \|_{2,0}) \leq Cc^{2}(1 + t)^{-1} \).

In order to obtain the estimates of the norms \( \| u_{xx}(t) \| \) and \( \| (\mathcal{U} u)_{x}(t) \| \) we use the smoothing operator \( \mathcal{F}(\varphi) = \cosh(\varphi) + i \sinh(\varphi) \mathcal{H} \) which was introduced in Section 2, where we take now \( \varphi(t, x) = \frac{1}{\varepsilon} \delta_{t}^{-1} |u(t, x)|^{2} \) and as in Section 2 we define \( \omega(t, x) = \frac{1}{\sqrt{\varepsilon}} |u(t, x)| \). Then applying Lemma 2.2 we obtain the energy type inequality for the functions \( h = u_{xx} \) and \( h = (\mathcal{U} u)_{x} \)

\[
\frac{d}{dt}(\mathcal{U} h \|^2 + \| \omega \mathcal{F} \sqrt{\partial_{x}^{2}} h \|^2) \leq 2\text{Im}(\mathcal{F} h, \mathcal{F}(\mathcal{N}_{u_{x}} h_{x} + \mathcal{N}_{u_{x}} \tilde{h}_{x})) + 2\text{Im}(\mathcal{F} h, \mathcal{F} R) + C\| \varphi \|_{\infty} (\| u \|_{\infty}^{4} + \| u \|_{1,0,\infty}^{2} + \| \varphi \|_{\infty}) \| h \|^2,
\]

where \( R = R_{1} \) or \( R = R_{2} \), respectively. Since \( R_{l} \) are bounded in \( \mathcal{L}^{2} \) we get via (3.2)

\[
\text{Im}(\mathcal{F} h, \mathcal{F} R_{l}) \leq C\| \varphi \|_{\infty} \| h \| \| R_{l} \| \leq Cc^{3}(1 + t)^{2\gamma}.
\]

To estimate the first summand \( \text{Im}(\mathcal{F} h, \mathcal{F}(\mathcal{N}_{u_{x}} h_{x} + \mathcal{N}_{u_{x}} \tilde{h}_{x})) \) in the right hand side of (3.5) we apply Lemma 2.3 to obtain

\[
\| (\mathcal{F} h, \mathcal{F}(\mathcal{N}_{u_{x}} h_{x} + \mathcal{N}_{u_{x}} \tilde{h}_{x})) \| \leq C\| \omega \mathcal{F} \sqrt{\partial_{x}^{2}} h \|^2 + Cc^{3}(1 + t)^{2\gamma}.
\]

Substitution of (3.6), (3.7) into (3.5) yields

\[
\frac{d}{dt}(\| u_{xxx} \|^2 + \| (\mathcal{U} u)_{x} \|^2)
\]

\[
+ (1 - Cc)(\| \omega \mathcal{F} \sqrt{\partial_{x}^{2}} u_{xx} \|^2 + \| \omega \mathcal{F} \sqrt{\partial_{x}^{2}} (\mathcal{U} u)_{x} \|^2)
\]

\[
\leq C(\| u_{xx} \|^2 + \| (\mathcal{U} u)_{x} \|^2) e^{2\| u \|_{\infty}^{2}} (\| u \|_{\infty}^{4} + \| u \|_{1,0,\infty}^{2} + \| \varphi \|_{\infty}).
\]
We also have
\[
\|\varphi_t\|_\infty = \left\| \partial_t \int_{-\infty}^{x} \left| u(t, x') \right|^2 dx' \right\|_\infty = \left\| \int_{-\infty}^{x} (u_t \bar{u} + \bar{u}_t u) dx \right\|_\infty
\]
\[
= \left\| \int_{-\infty}^{x} ((u_{xx} - N') \bar{u} - (\bar{u}_{xx} - N') u) dx \right\|_\infty
\]
\[
= \left\| u_x \bar{u} - \bar{u}_x u - \int_{-\infty}^{x} (N' \bar{u} - \bar{N} u) dx \right\|_\infty \leq C \| u \|_{1, 0, \infty}^2 (1 + \| u \|^2).
\]
Therefore from (3.8) and the Gronwall inequality we get \( \| u_{xx} \|^2 + \| (\mathcal{H}u)_x \|^2 < 2e^2(1 + t)^2 \). Thus we have the estimate
\[
(3.9) \quad \| u(t) \|_Y < 3e(1 + t)^{\gamma}
\]
for all \( t \in [0, T] \).

To prove the estimate \( \| u(t) \| + \sqrt{1 + t} \| u(t) \|_{1, 0, \infty} < \sqrt{\varepsilon} \), as in [7] we change the dependent variable \( u(t, x) = \frac{1}{\sqrt{t}} Ev(\tau, \chi) \), where \( \chi = \frac{x}{2\tau}, \quad \tau = t + 1, \quad E = e^{ix^2} \).

Then since \( u_x(t, x) = \frac{1}{\sqrt{t}} e^{ix^2} \mathcal{H}v(\tau, \chi) \) with \( \mathcal{H} = i\chi + \frac{1}{2} \partial_\chi \), we write (1.1) in the form
\[
(3.10) \quad \begin{cases}
iv_\tau + \frac{1}{4\tau^2} v_\chi = \frac{1}{\tau} \mathcal{M}(v), & \chi \in \mathbb{R}, \quad \tau \geq 1, \\
v(1, \chi) = u(0, \chi)e^{-ix^2}, & \chi \in \mathbb{R},
\end{cases}
\]
where \( \mathcal{M}(v) = ia(2|v|^2 \mathcal{H}v + v^2 \mathcal{H}v) + 3\lambda_1 E^2 v^2 \mathcal{H}v + \lambda_2 E^2 (2|v|^2 \mathcal{H}v + v^2 \mathcal{H}v) + 3\lambda_3 E^4 v^2 \mathcal{H}v. \) Note that \( \mathcal{H}u(t, x) = \frac{i}{\sqrt{t}} Ev(\tau, \chi) \), therefore we have the following relations \( \| v_\chi \| + \| \mathcal{H}v_\chi \| \leq C \| \mathcal{H}u \|_{1, 0} \leq C e^{\tau \gamma}, \quad \| v_\chi \| = \sqrt{2} \| u_\chi \| \) and \( \| \mathcal{H}^2 v \| = \sqrt{2} \| u_{xx} \| \leq C e^{\tau \gamma}. \) Since \( \sqrt{1 + \| u \|_{1, 0, \infty} \leq \| v \|_{\infty} + \| \mathcal{H}v \|_{\infty} \) we need to prove the estimate \( \| v \| + \| v \|_{\infty} + \| \mathcal{H}v \|_{\infty} < \sqrt{\varepsilon}. \) Using the identity \( \partial_\chi \mathcal{H}v = \frac{1}{2} (v \mathcal{H}v - v \mathcal{H}v) + \frac{1}{4\tau} (|v|^2)_\chi \),
we write the following representations \( 2ia|v|^2 \mathcal{H}v = 2av \text{Im}(v \overline{\mathcal{H}v}) + \frac{ia}{2\tau} v(|v|^2)_\chi \) and \( iav^2 \overline{\mathcal{H}v} = -av \text{Im}(v \overline{\mathcal{H}v}) + \frac{ia}{4\tau} v(|v|^2)_\chi. \) Therefore we can represent the nonlinearity in the form \( \mathcal{M}(v) = -gv + Z_0 + P_0 \), where \( g = -a \text{Im}(v \overline{\mathcal{H}v}) = -a\chi |v|^2 - \frac{a}{2\tau} (|v|^2)_\chi \) is a real valued function
\[
Z_0 = i\chi (3\lambda_1 E^2 v^3 - \lambda_2 E^2 v^5 - 3\lambda_3 E^4 v^3).
\]
and

\[ P_0 = \frac{3ia}{4\tau} v(|v|^2)_x + \frac{1}{2\tau}(3\lambda_1 E^2 v^2 v_x + \lambda_2 E^2 (2|v|^2\dot{v}_x + \ddot{v}^2 v_x) + 3\lambda_3 E^4 \dddot{v}^2 v_x). \]

Analogously multiplying equation (3.10) by the operator \( \mathcal{A} \) and using the identity \( \mathcal{A}(\psi\phi\dot{\psi}) = \psi\dot{\phi}\mathcal{A}\psi + \phi\dot{\psi}\mathcal{A}\psi + \phi\psi\ddot{\mathcal{A}}\psi \) we represent the nonlinearity in the following form \( \mathcal{A}\mathcal{N}(v) = ia\mathcal{A}(2|v|^2 \mathcal{A}v + v^2 \mathcal{A}v) + \frac{1}{2\tau}(3\lambda_1 E^2 (v^2 \mathcal{A}^2 v + 2v(\mathcal{A}v)^2) + \lambda_2 E^2 (\dot{v}^2 \mathcal{A}^2 v + 4v|\mathcal{A}v|^2 + 2v(\mathcal{A}v)^2) + 2|v|^2 \mathcal{A}^2 v + 3\lambda_3 E^4 (2v(\mathcal{A}v)^2 + \dddot{v}^2 \mathcal{A}^2 v) = -g\mathcal{A}v + Z_1 + P_1, \) where

\[ Z_1 = i\chi(9\lambda_1 E^2 v^2 \mathcal{A}v + \lambda_2 E^2 \dddot{v}^2 \mathcal{A}v - 9\lambda_3 E^4 \dddot{v}^2 \overline{\mathcal{A}v}) \]

and

\[ P_1 = \frac{a}{2\tau} v \text{Im}(v\overline{\mathcal{A}v})_x + \frac{3ia}{4\tau} \mathcal{A}(v(|v|^2)_x) + \frac{3\lambda_1}{2\tau} E^2 (v^2 \mathcal{A}v)_x + \frac{\lambda_2}{2\tau} E^2 (2|v|^2 (\mathcal{A}v)_x - 4v\mathcal{A}v\dot{v}_x + 2v\overline{\mathcal{A}v}\dot{\phi}_x + \ddot{v}^2 (\mathcal{A}v)_x + 6v_x\overline{v}\mathcal{A}v) + \frac{3\lambda_3}{2\tau} E^4 (\overline{v}^2 \overline{\mathcal{A}v})_x. \]

Via the inequality (3.2) we have \( \|g\|_\infty \leq Ce^2. \) Taking into account the commutator relation \([\mathcal{A}, \mathcal{A}] = 0, \) where \( \mathcal{A} = i\partial_\tau + \frac{1}{4\tau^2} \partial^2_\chi, \) we get from (3.10)

\[ (3.11) \quad \mathcal{C}\mathcal{A}^j v = \frac{1}{\tau} (-g\mathcal{A}^j v + Z_j + P_j), \quad j = 0, 1. \]

We define the evolution operator \( \mathcal{V}(\tau)\phi = \mathcal{F}^{-1} e^{it\chi^2/4\tau^2} \phi = \frac{\sqrt{\tau}}{\sqrt{\pi}} \int e^{-iT(\chi-y)^2} \phi(y)dy. \) First of all we note that \( \|\mathcal{V}(\tau)\phi\| = \|\phi\|. \) Also it is easy to see that the estimates \( \|\mathcal{V}(\tau)\phi\|_\infty \leq \sqrt{\tau} \|\phi\|, \quad \|\mathcal{V}(\tau) - 1\phi\|_\infty \leq \|e^{it\chi^2/4\tau^2} - 1\phi(\xi)\|_1 \leq C\tau^{-\alpha} \|\phi\|_{1,0} \) and analogously \( \|\mathcal{V}(\tau) - 1\phi\| \leq C\tau^{-2\alpha} \|\phi\|_{1,0} \) are valid, where \( \alpha \in [0, \frac{1}{4}). \) Note that

\[ \mathcal{V}(-\tau)(E^{\alpha-1}\phi) = D_\alpha E^{\alpha(\alpha-1)}\mathcal{V}(-\omega\tau)\phi = D_\alpha E^{\alpha(\alpha-1)}\phi + D_\alpha E^{\alpha(\alpha-1)}(\mathcal{V}(-\omega\tau) - 1)\phi, \]

where \( D_\alpha\phi(\chi) = \frac{1}{\sqrt{\alpha}} \phi' \left( \frac{\chi}{\alpha} \right) \), we take here \( \alpha = 3, -1, 3. \) Multiplying (3.11) by \( \mathcal{V}(-\tau) \) we obtain for the function \( y_j = \mathcal{V}(-\tau)\chi^j v \)

\[ (3.12) \quad i\partial_\tau y_j = \frac{1}{\tau} (-g y_j + Z_j + G_j), \quad j = 0, 1, \]
where
\[
\dot{Z}_0 = 3i\lambda_1 \mathcal{D}_3 E^6(\chi y_0^3) - i\lambda_2 \mathcal{D}_{-1} E^2(\chi y_0 y_0^2) - 3i\lambda_3 \mathcal{D}_{-3} E^{12}(\chi y_0^3),
\]
\[
\dot{Z}_1 = 9i\lambda_1 \mathcal{D}_3 E^6(\chi y_0^2 y_1) + i\lambda_2 \mathcal{D}_{-1} E^2(\chi y_1 y_0^2) - 9i\lambda_3 \mathcal{D}_{-3} E^{12}(\chi y_0^2 y_1),
\]
and the remainder terms are
\[
G_0 = (\mathcal{V}(-\tau) - 1)gv + g(1 - \mathcal{V}(-\tau))v + \mathcal{V}(-\tau)P_0 + 3i\lambda_1 \mathcal{D}_3 E^6((\mathcal{V}(\tau) - 1)\chi v^3 + \chi(v^3 - y_0^3))
\]
\[- i\lambda_2 \mathcal{D}_{-1} E^2((\mathcal{V}(-3\tau) - 1)\chi v^2 + \chi(v^2 - y_0^2))
\]
\[- 3i\lambda_3 \mathcal{D}_{-3} E^{12}((\mathcal{V}(-5\tau) - 1)\chi v^3 + \chi(v^3 - y_0^3))
\]
and
\[
G_1 = (\mathcal{V}(-\tau) - 1)g\mathcal{V}v + g(1 - \mathcal{V}(-\tau))\mathcal{V}v + \mathcal{V}(-\tau)P_1 + 9i\lambda_1 \mathcal{D}_3 E^6((\mathcal{V}(\tau) - 1)\chi v^2 \mathcal{V}v + \chi(v^2 \mathcal{V}v - y_0^2))
\]
\[+ i\lambda_2 \mathcal{D}_{-1} E^2((\mathcal{V}(-3\tau) - 1)\chi v^2 \mathcal{V}v + \chi(v^2 \mathcal{V}v - y_0^2))
\]
\[- 9i\lambda_3 \mathcal{D}_{-3} E^{12}((\mathcal{V}(-5\tau) - 1)\chi v^2 \mathcal{V}v + \chi(v^2 \mathcal{V}v - y_0^2)).
\]
Via the inequality (3.2) we have the estimate \( ||G_j||_\infty + ||G_j|| \leq C\epsilon^3 \tau^{3y-1/4}, \) \( j = 0, 1. \) Now we change the dependent variable in equation (3.12) as follows
\[
y_j = w_j(\tau, \chi)A(\tau, \chi), \text{ where } A(\tau, \chi) = \exp(i \int_{s}^{\tau} g(s, \chi) \frac{ds}{s}) \text{ to get}
\]
\[(3.13) \quad i\partial_\tau w_j + \frac{1}{\tau} \overline{A}(\dot{Z}_j + G_j) = 0, \quad j = 0, 1.
\]
Integration of (3.13) with respect to time \( \tau \) yields
\[
w_j(\tau) - w_j(s) = \int_s^\tau \overline{A} \dot{Z}_j \frac{d\tau}{\tau} + O\left(\epsilon^3 \int_s^\tau \tau^{3y-5/4} d\tau\right).
\]
By virtue of the identity \( e^{i\omega x^2} = (1 + i\omega x^2)^{-1} \partial_x (\tau e^{i\omega x^2}) \) we integrate by parts in the integral of the form
\[
\int_s^\tau \overline{A}(\tau) e^{i\omega x^2} \chi \phi(\tau, \chi) \frac{d\tau}{\tau} = \frac{\overline{A} e^{i\omega x^2} \chi \phi}{1 + i\omega x^2} \Bigg|_s^\tau
\]
\[+ \int_s^\tau \frac{\chi}{1 + i\omega x^2} \left( \frac{i\omega x^2}{1 + i\omega x^2} \phi + \phi - \tau \phi_x + i\phi_x \right) d\tau,
\]
whence via the estimate \( \sup_{x \in \mathbb{R}} \left| \frac{x}{1 + i\omega x^2} \right| \leq |\omega|^{-1/2} \) we obtain

\[
(3.14) \quad \left\| \int_{s}^{\tau} \bar{A}(\tau) e^{i\omega x^2} \chi \varphi(x, \chi) \frac{d\tau}{\tau} \right\| \leq C s^{-\mu} \left( \| \varphi \|_{p} + \sup_{\tau \geq 1} \| \tau^{1-\gamma} \| \varphi_{\tau} \|_{p} \right),
\]

where \( \tau > s, \ p = 2, \infty, \ \mu = \frac{1}{4} - 3\gamma > 0, \ \omega \in \mathbb{R} \setminus 0 \). From equation (3.12) we have a rough estimate \( \| \partial_{t} y_{j} \|_{\infty} \leq C \tau \epsilon^{-1}, \ j = 0, 1 \). Using (3.14) with \( \omega = \frac{2}{3}, 2\frac{2}{3} \) and \( \varphi = \mathscr{D}_{3}(\gamma_{0}^{2} y_{j}), \ \mathscr{D}_{-1}(\gamma_{0}^{2} y_{j}), \ \mathscr{D}_{3}(\gamma_{0}^{2} y_{j}) \), respectively we get

\[
(3.15) \quad \| w_{j}(\tau) - w_{j}(s) \|_{p} \leq C e^{3s^{-\mu}},
\]

for all \( 1 < s < \tau \), where \( p = 2, \infty, \ j = 0, 1, \ \mu = \frac{1}{4} - 3\gamma > 0 \). By virtue of (3.15) we obtain \( \sup_{\tau \in [1, \tau + 1]} \| w_{j}(\tau) \|_{\infty} < C \epsilon \). Therefore in view of estimate (3.2) we find \( \| u \| + \sqrt{1 + \tau} \| u \|_{1,0,\infty} \leq \| v \| + \| v \|_{\infty} + \| \mathscr{X} v \|_{\infty} \leq \| (1 - \gamma^{(-\tau)}) v \|_{\infty} + \| (1 - \gamma^{(-\tau)}) \mathscr{X} v \|_{\infty} + \| w_{0} \| + \| w_{0} \|_{\infty} + \| w_{1} \|_{\infty} \leq C \tau \epsilon^{-\gamma} + C \epsilon < \sqrt{\epsilon} \). The contradiction obtained gives us the result of the lemma. \( \square \)

§ 4. Proof of Theorems 1.1–1.2

Proof of Theorem 1.1. We have the existence result and estimate (1.2) by Lemma 3.1. Via inequality (3.15) there exist unique limits \( w_{j}^{+} \in L^{\infty} \cap L^{2} \) such that \( \lim_{t \to \infty} w_{j}(t) = w_{j}^{+} \) in \( L^{\infty} \cap L^{2} \). Hence there exists a unique limit \( g^{+} = \lim_{t \to \infty} g(t) = \text{Im}(w_{0}^{+} \overline{\bar{w}}_{1}^{+}) \) in \( L^{\infty} \). Thus we get

\[
(4.1) \quad \partial_{t}^{j} u(t, x) = \frac{1}{\sqrt{\tau}} e^{i\tau x^{2}} \mathscr{X}^{j} v = \frac{1}{\sqrt{\tau}} e^{i\tau x^{2}} \gamma \mathscr{X}^{j} v + \frac{1}{\sqrt{\tau}} e^{i\tau x^{2}} (1 - \gamma \mathscr{X}^{j} v
\]

\[
= \frac{1}{\sqrt{\tau}} e^{i\tau x^{2}} w_{j}(\tau, \frac{x}{2\tau}) \exp \left( i \int_{1}^{\tau} g(s, \frac{x}{2\tau}) \frac{ds}{s} \right) + O(\tau^{-1/2-\mu})
\]

\[
= \frac{1}{\sqrt{t}} e^{i\tau x^{2}} w_{j}(\tau, \frac{x}{2\tau}) \exp \left( i \int_{1}^{\tau} g(s, \frac{x}{2\tau}) \frac{dt}{t} \right) + O(\tau^{-1/2-\mu})
\]

uniformly with respect to \( x \in \mathbb{R} \), here \( \mu = \frac{1}{4} - 3\gamma > 0 \). For the phase of the asymptotic representation (4.1) we write the identity \( \int_{1}^{\tau} g(s) \frac{ds}{s} = g^{+} \log \tau + \Phi(\tau) \),

where \( \Phi(\tau) = (g(\tau) - g^{+}) \log \tau + \int_{1}^{\tau} (g(s) - g(\tau)) \frac{ds}{s} \). We have

\[
(4.2) \quad \Phi(\tau) - \Phi(s) = \int_{s}^{\tau} (g(t) - g(\tau)) \frac{dt}{t} + (g(\tau) - g^{+}) \log \frac{\tau}{s},
\]
for all $1 < s < \tau$. Applying estimates (3.1) of Lemma 3.1 and inequality (3.15) to (4.2) we get $\|\Phi(\tau) - \Phi(s)\|_{\infty} \leq C e^{-\mu}$ for $1 < s < \tau$. This implies that there exists a unique limit $\Phi^+ = \lim_{t \to \infty} \Phi(t) \in L^\infty$ such that

$$
\|\Phi(t) - \Phi^+\|_{\infty} \leq Ct^{-\mu}.
$$

By virtue of (4.3) we find

$$
\|\int_1^t g(\tau) \frac{d\tau}{\tau} - g^+ \log t - \Phi^+\|_{\infty} \leq C \epsilon t^{-\mu}.
$$

We now put $u_j^+ = w_j^+ \exp(i\Phi^+)$. Then we obtain the asymptotics (1.4) for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$. Via (4.4) and (3.15) we have

$$
\|\mathcal{N}^j v - u_j^+ \exp(ig^+ \log t)\| \leq \|\mathcal{N}(-t)\mathcal{N}^j v - u_j^+ e^{ig^+ \log t}\| + \|(1 - \mathcal{N}(-t))\mathcal{N}^j v\|
$$

\leq \left\| w_j \exp\left(i \int_1^t g(\tau) \frac{d\tau}{\tau}\right) - w_j^+ \exp(ig^+ \log t + i\Phi^+) \right\| + O(t^{-\mu}) = O(t^{-\mu}),

whence we get

$$
\|\mathcal{U}(-t) \partial_t^j u(t) - \mathcal{F}^{-1}(u_j^+ \exp(ig^+ \log t))\|
$$

\leq \|\mathcal{U}(-t) \partial_t^j u(t) - \mathcal{U}(-t) - \sqrt{-2i} \mathcal{N} \mathcal{H} \mathcal{N}^j v\|

+ \sqrt{2} \|\mathcal{U}(-t) \mathcal{H} \mathcal{F} \mathcal{M} - 1 \mathcal{F}^{-1} \mathcal{N}^j v\|

+ \sqrt{2} \|\mathcal{F}^{-1}(\mathcal{N}^j v - u_j^+ \exp(ig^+ \log t))\| = O(t^{-\mu}),

whence (1.4) follows. This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** We prove the theorem by contradiction, so we assume that $u^+$ is not identically zero. Multiplying equation (1.1) by $\mathcal{U}(-t)$ and integrating with respect to time we find

$$
\mathcal{U}(-t)u(t) - \mathcal{U}(-s)u(s) = -i \int_s^t \mathcal{U}(-\tau) \mathcal{N}(u) d\tau.
$$

We decompose the nonlinear term $\mathcal{N}(u)$ as follows

$$
\mathcal{N}(u) = \mathcal{N}(u) - \mathcal{N}(\mathcal{U}(t) u^+))
$$

$$
+ (\mathcal{N}(\mathcal{U}(t) u^+) - \mathcal{N}(\mathcal{U}(t) \mathcal{M}u^+)) + \mathcal{N}(\mathcal{H} \mathcal{N} u^+),
$$

where we have used the identity $\mathcal{U}(t) \mathcal{M} = \mathcal{H} \mathcal{F}$.

Since $\|\mathcal{N}(\mathcal{H} \mathcal{N} u^+)\| \geq \frac{1}{t} \|\alpha \xi |\mathcal{H} \mathcal{N} u^+|^2 |u^+| - \frac{C}{t^3} \|u^+\|_{1,\delta}^3$, we have by (4.5), (4.6)
and estimates for the solution $u(t)$ provided by Theorem 1.1
\[
\|\mathcal{U}(-t)u(t) - \mathcal{U}(-s)u(s)\| \geq \|a_\xi|\hat{u^+}|^2\hat{u^+}\| \int_s^t \frac{d\tau}{\tau} - C\|u^+\|_{1,\delta}^2 \int_s^t \frac{d\tau}{\tau^2}
\]
\[
- C(\varepsilon^2 + \|u^+\|_{1,\delta}^2) \int_s^t (\|u(\tau) - \mathcal{U}(\tau)u^+\|_{1,0} + \|(\mathcal{M} - 1)u^+\|_{1,0}) \frac{d\tau}{\tau}.
\]
This implies that for any small number $\theta > 0$ there exists a time $T(\theta)$ such that for any $t > s > T(\theta)$
\[
\|\mathcal{U}(-t)u(t) - \mathcal{U}(-s)u(s)\| \geq (\|a_\xi|\hat{u^+}|^2\hat{u^+}\| - \theta) \int_s^t \frac{d\tau}{\tau}
\]
which means $u^+ = 0$. Thus we get the desired contradiction. If the solution satisfies the conservation of the $L^2$ norm, we have also $u \equiv 0$. Theorem 1.2 is proved. $\square$

References


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