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On the Differential Equation $x^{(m)} = f(t, x)$ in Banach Spaces

By

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1. Introduction

Assume that $I = [0, a]$, E is a Banach space, $B = \{x \in E : \|x\| \le b\}$ and $f : I \times B \mapsto E$ is a bounded continuous function. Let α be the Kuratowski measure of noncompactness in E (cf. [2]). A. Cellina [3] proved the following theorem:

Suppose that f satisfies the condition

$$\alpha(f(I \times X)) \le w(\alpha(X)) \quad \text{for } X \subset B,$$

where $w : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a continuous nondecreasing function such that $w(0) = 0$ and $\int_{0+}(dr/w(r)) = \infty$. Then there exists a solution of the Cauchy problem

$$x' = f(t, x), \quad x(0) = 0.$$

In this paper we shall extend this result to the differential equation

$$x^{(m)} = f(t, x).$$

2. A differential inequality

Let $m \ge 1$ be a natural number. Using the argument from [1; p. 97–98] we can prove the following

Lemma 1. *Let $w : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a continuous nondecreasing function and let $g : [0, c) \to \boldsymbol{R}_+$ be a C^m function satisfying the inequalities*

$$g^{(j)}(t) \ge 0, \quad j = 0, 1, \dots, m$$

$$g^{(j)}(0) = 0, \quad j = 0, 1, \dots, m-1$$

$$g^{(m)}(t) \le w(g(t)), \quad t \in [0, c).$$

Then

$$g^{(m-1)}(t) \le w_0(g(t)), \quad \text{for } t \in [0, c),$$

where $w_0(r) = 2^{a_m} r^{1/m}[w(r)]^{1-1/m}$ and a_m is a positive number.

Lemma 2. *Let functions w and g satisfy the assumptions of Lemma* 1. *If* $w(0) = 0$ *and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty,$$

then $g = 0$.

Proof. By Lemma 1 we have

$$g^{(m-1)}(t) \le w_0(g(t)), \quad \text{for } t \in [0, c),$$

where $w_0(r) = 2^{a_m} r^{1/m}[w(r)]^{1-1/m}$. Let us remark that $w_0(0) = 0$ and

$$\int_{0+} \frac{dr}{\sqrt[m-1]{r^{m-2}w_0(r)}} = 2^{-a_m/m-1} \int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty.$$

Repeating the above argument successively for $m-1, m-2, \dots, 2$, we deduce that there exists a function $h: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that $h(0) = 0$, $\int_{0+}(dr/h(r)) = \infty$ and $g'(t) \le h(g(t))$ for $t \in [0, c)$. As $g(0) = 0$, this implies that $g(t) = 0$ for $t \in [0, c)$.

3. The main result

Consider the Cauchy problem

$$(1) \qquad \begin{aligned} x^{(m)} &= f(t, x) \\ x(0) = 0, \quad x'(0) &= \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1}, \end{aligned}$$

where $m \ge 1$ and $\eta_1, \dots, \eta_{m-1} \in E$.

Theorem. *Let* $w: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *be a continuous nondecreasing function such that* $w(0) = 0$ *and*

$$(2) \qquad \int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty.$$

If

$$(3) \qquad \alpha(f(t, X)) \le w(\alpha(X)) \quad \textit{for } t \in I \quad \textit{and } X \subset B,$$

then there exists an interval $J = [0, d]$ *such that the problem* (1) *has at least one solution defined on* J.

Proof. Put $M = \sup\{\|f(t, x)\| : t \in I, x \in B\}$. Choose a positive number d such that $d \le a$ and

$$(4) \qquad \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M\frac{d^m}{m!} \le b.$$

Let $C = C(J, E)$ be the Banach space of continuous functions $J \to E$ with usual supremum norm $\|\cdot\|_C$, where $J = [0, d]$, and let $\tilde{B} \subset C$ be the subset of those functions with values in B. The problem (1) is equivalent to the integral equation

$$x(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, x(s))\,ds \quad (t \in J), \tag{5}$$

where $p(t) = \sum_{j=1}^{m-1} \eta_j(t^j/j!)$. We define a mapping F by

$$F(x)(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, x(s))\,ds \quad (t \in J, x \in \tilde{B}).$$

It is known (cf. [5]) that F is a continuous mapping $\tilde{B} \to \tilde{B}$ and the set $F(\tilde{B})$ is equicontinuous. For any positive integer n put

$$v_n(t) = \begin{cases} p(t) & \text{if } 0 \le t \le d/n \\ p(t) + \dfrac{1}{(m-1)!}\displaystyle\int_0^{t-d/n} (t-s)^{m-1} f(s, v_n(s))\,ds & \text{if } d/n \le t \le d. \end{cases}$$

Then, by (4), $v_n \in \tilde{B}$ and

$$\lim_{n\to\infty} \|v_n - F(v_n)\|_C = 0. \tag{6}$$

Put $V = \{v_n : n \in N\}$ and $Z(t) = \{x(t) : x \in Z\}$ for $t \in J$ and $Z \subset C$. As $V \subset \{v_n - F(v_n) : n \in N\} + F(V)$ and $V \subset \tilde{B}$, from (6) it follows that the set V is equicontinuous and the function $t \mapsto v(t) = \alpha(V(t))$ is continuous on J. Denote by H a separable Banach subspace of E such that

$$v_n(s), f(s, v_n(s)) \in H \quad \text{for } s \in J \quad \text{and} \quad n \in N.$$

Let β_H be the Hausdorff measure of noncompactness in H. It follows from (3) that $\beta_H(f(s, V(s))) \le \alpha(f(s, V(s))) \le w(\alpha(V(s)))$.

Applying now the Mönch lemma [4], we get

$$\begin{aligned}
\alpha(F(V)(t)) &= \alpha\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, v_n(s))\,ds : n \in N\right\}\right) \\
&\le 2\beta_H\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, v_n(s))\,ds : n \in N\right\}\right) \\
&\le \frac{2}{(m-1)!}\int_0^t \beta_H(\{(t-s)^{m-1} f(s, v_n(s)) : n \in N\})\,ds \\
&\le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} \beta_H(f(s, V(s)))\,ds \\
&\le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(\alpha(V(s)))\,ds.
\end{aligned}$$

On the other hand, from (6) and the inclusion

$$V(t) \subset \{v_n(t) - F(v_n)(t) : n \in N\} + F(V)(t)$$

it follows that $v(t) \leq \alpha(F(V)(t))$. Hence

$$v(t) \leq \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s))\, ds \quad \text{for } t \in J.$$

Putting $g(t) = (2/(m-1)!) \int_0^t (t-s)^{m-1} w(v(s))\, ds$, we see that $g \in C^m$, $v(t) \leq g(t)$, $g^{(j)}(t) \geq 0$ for $j = 0, 1, \ldots, m$, $g^{(j)}(0) = 0$ for $j = 0, 1, \ldots, m-1$ and $g^{(m)}(t) = 2w(v(t)) \leq 2w(g(t))$ for $t \in J$.

Moreover, by (2),

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1} 2w(r)}} = \infty.$$

By Lemma 2 from this we deduce that $g(t) = 0$ for $t \in J$. Thus $\alpha(V(t)) = 0$ for $t \in J$. Therefore for each $t \in J$ the set $V(t)$ is relatively compact in E, and by Ascoli's theorem the set V is relatively compact in C. Hence we can find a subsequence (v_{n_k}) of (v_n) which converges in C to a limit x. As F is continuous, from (6) we conclude that $x = F(x)$, so that x is a solution of (5).

4. An Example

Consider the function $w(r) = r|\ln r|^m$ for $0 < r \leq e^{-m}$, $w(0) = 0$. It can be easily verified that w is continuous, nondecreasing and

$$|w(\xi) - w(\eta)| \leq w(|\xi - \eta|) \quad \text{for } 0 \leq \xi, \quad \eta \leq e^{-m}. \tag{7}$$

Moreover,

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1} w(r)}} = \int_{0+} \frac{dr}{r|\ln r|} = \infty.$$

Let $E = C(0,1)$ and $B = \{x \in E : \|x\| \leq e^{-m}/2\}$. We define a function $f_1 : B \to E$ by

$$f_1(x)(\tau) = w(|x(\tau)|) \quad \text{for } \tau \in [0,1] \text{ and } x \in B.$$

By (7) we get $\|f_1(x) - f_1(y)\| \leq w(\|x - y\|)$ for $x, y \in B$.

From this we deduce that for a given completely continuous function $f_2 : B \to E$ the function $f = f_1 + f_2$ satisfies the inequality

$$\alpha(f(X)) \leq w(\alpha(X)) \text{ for } X \subset B.$$

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