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Application of the Trudinger-Moser Inequality to a Parabolic System of Chemotaxis

By

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Dedicated to Professor Kyûya Masuda on the occasion of his 60th birthday

1. Introduction

In 1970 Keller and Segel [15] proposed a mathematical model describing chemotactic aggregation of cellular slime molds which move preferentially towards relatively high concentrations of a chemical secreted by the amoebae themselves. With the cell density of the cellular slime molds $u(x,t)$ and the concentration of the chemical substance $v(x,t)$ at place x and time t, a prototype of the Keller-Segel models is described as the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (d_1 \nabla u - a_1 u \nabla v) & \text{in } \Omega, \quad t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - a_2 v + a_3 u & \text{in } \Omega, \quad t > 0, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, d_1, d_2, a_1, a_2, a_3 are positive numbers. The boundary conditions on u and v are homogeneous Neumann conditions on $\partial\Omega$.

To study the Keller-Segel system, it is convenient to transform the system by

$$d_1 t \mapsto t, \quad \frac{a_1}{d_1} = \chi, \quad \frac{d_1}{d_2} = \varepsilon, \quad \frac{a_2}{d_2} = \gamma, \quad \frac{a_3}{d_2} = \alpha.$$

Then we arrive at the following initial-boundary value problem

$$\text{(KS)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \Omega, \quad t > 0, \\ \varepsilon \dfrac{\partial v}{\partial t} = \Delta v - \gamma v + \alpha u & \text{in } \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{on } \Omega. \end{cases}$$

Here, u_0 and v_0 are non-negative smooth functions on Ω.

One of interesting aspects of the Keller-Segel model (KS) is the possibility of blow-up of solutions in finite time (see Nanjundiah [21]). Especially in two space dimensions, a conjecture by Childress [7] and Childress and Percus [8] states that there exists a threshold number c such that if $\int_\Omega u_0(x)dx < c$ then the solution (u, v) exists globally in time, and if $\int_\Omega u_0(x)dx > c$ then $u(x, t)$ can form a delta function singularity in finite time. We refer to such a blowup phenomenon as chemotactic collapse. In the case of radial initial functions (u_0, v_0) on $\Omega = \{x \in \boldsymbol{R}^2; |x| < L\}$, the threshold number is conjectured as $c = 8\pi/(\alpha\chi)$.

The conjecture mentioned above is strongly supported by results in [11, 14, 19], where they considered the simplified system by setting $\varepsilon = 0$ in (KS). Jäger and Luckhaus [14] showed the global existence of solutions in time when the initial functions have small enough mass, and that there exist radial solutions which blow up at the origin in finite time. Nagai [19] confirmed that the possibility of blow-up of radial solutions to (KS) with $\varepsilon = 0$ requires the threshold number $8\pi/(\alpha\chi)$ for radial functions u_0 on $\Omega = \{x \in \boldsymbol{R}^2; |x| < L\}$ as follows:

(i) If the radial function u_0 satisfies the condition $\int_\Omega u_0(x)dx < 8\pi/(\alpha\chi)$, then the radial solution (u, v) exists globally in time and is globally bounded;

(ii) If the radial function u_0 satisfies the condition such that $\int_\Omega u_0(x)dx > 8\pi/(\alpha\chi)$ and $\int_\Omega u_0(x)|x|^2 dx$ is sufficiently small, then the radial solution (u, v) blows up in finite time.

Concerning chemotactic collapse, Herrero and Velázquez [11] succeeded to construct radial solutions to (KS) with $\varepsilon = 0$ on $\Omega = \{x \in \boldsymbol{R}^2; |x| < L\}$ such that $u(x, t)$ blows up in finite time to form a δ-function singularity at the origin by using matched asymptotic expansions methods.

As for (KS) with $\varepsilon > 0$ in two space dimensions, Yagi [28] has studied the local existence of solutions and some norm behavior of maximal solutions to more general parabolic systems including the Keller-Segel model, and showed the global existence of solutions under the condition such that $\int_\Omega u_0(x)dx$ is sufficiently small. Recently, Herrero and Velázquez [12, 13] have extended their results in [11] to the case $\varepsilon > 0$ in (KS) by using the similar method to that in [11].

For related results to the chemotaxis system, we refer to [9, 17, 20, 23, 24]. We also mention that a system similar to (KS) with $\varepsilon = 0$ arises from another mathematical model describing the gravitational interaction of particles (see Biler and Nadzieja [2], Biler [4] and the references therein), and in [2, 4] the existence of global solutions in time and blow-up of solutions have been studied.

In this paper we prove the time global existence and L^∞ estimate of the solution to (KS) with $\varepsilon > 0$ in the case $\Omega \subset \boldsymbol{R}^2$ by making use of the Trudinger-Moser inequality extended to the Sobolev space $W^{1,p}(\Omega)$ which is stated in Section 2. As usual we denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$ for nonnegative integer k, and $H^{k+\theta}(\Omega)$ by the intermediate space between $H^k(\Omega)$ and H^{k+1} for any $0 < \theta < 1$ (see [16, 26]). Our main theorem is the following

Theorem 1.1. *Let Ω be a bounded domain with smooth boundary in $\boldsymbol{R}^2$. Assume $u_0, v_0 \in H^{1+\varepsilon_0}(\Omega)$ for some $0 < \varepsilon_0 \leqq 1$, and $u_0 \geqq 0$, $v_0 \geqq 0$ on Ω.*

(i) *If $\int_\Omega u_0(x)dx < 4\pi/(\alpha\chi)$, then* (KS) *admits a unique classical solution (u, v) on $\bar{\Omega} \times (0, \infty)$ satisfying*

$$\sup_{t \geqq 0}\{\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty}\} < \infty.$$

(ii) *Let $\Omega = \{x \in \boldsymbol{R}^2; |x| < L\}$ and (u_0, v_0) be radial in x. Then the same assertion as* (i) *holds under the condition $\int_\Omega u_0(x)dx < 8\pi/(\alpha\chi)$.*

Our theorem and the results of Herrero and Velázquez [12, 13] show that under radial conditions on the initial functions the value $8\pi/(\alpha\chi)$ is the threshold number conjectured by Childress [7] and Childress and Percus [8] as to whether the solution to (KS) exists globally in time or not.

2. The Trudinger-Moser inequality

Let Ω be a domain in $\boldsymbol{R}^n (n \geqq 2)$ with finite measure $|\Omega| = \int_\Omega dx < \infty$ and $W_0^{1,p}(\Omega)$ denotes the Banach space obtained from C^1-functions $u(x)$ with compact support in Ω by completion with the norm

$$\|\nabla u\|_{L^p(\Omega)} (\text{or simply } \|\nabla u\|_p) = \left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p}.$$

Moser showed a sharp form on an inequality by Trudinger [27] as the following

Theorem ([18]). *Let $u \in W_0^{1,n}(\Omega), n \geqq 2$ and*

$$\int_\Omega |\nabla u|^n\, dx \leqq 1.$$

Then there exists a constant c depending only on n such that

$$\int_\Omega e^{\alpha|u|^p}\, dx \leqq c|\Omega|,$$

where

$$p = \frac{n}{n-1}, \quad \alpha \leqq \alpha_n = n\omega_{n-1}^{1/(n-1)},$$

and ω_{n-1} *is the* $(n-1)$*-dimensional surface area of the unit sphere in* $\boldsymbol{R}^n$. *The integral on the left actually is finite for any positive* $\alpha \leqq \alpha_n$, *but if* $\alpha > \alpha_n$ *it can be made arbitrarily large by an appropriate choice of* u.

As a corollary to this theorem the following inequality holds.

Corollary. *Let* $u \in W_0^{1,n}(\Omega)$ *and* $n \geqq 2$. *Then there exists a constant* c *depending only on* n *such that*

$$\int_\Omega \exp|u| dx \leqq c|\Omega| \exp\left(\frac{1}{\beta_n}\|\nabla u\|_n^n\right),$$

where

$$\beta_n = n\left(\frac{n\alpha_n}{n-1}\right)^{n-1}.$$

Proof. When $\|\nabla u\|_n = 0$, the proof is trivial. So we consider only the case $\|\nabla u\|_n \neq 0$. Put $v = u/\|\nabla u\|_n$ and $q = n$. Then we have

$$\int_\Omega |\nabla v|^n dx = 1 \qquad \text{and} \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

where $p = n/(n-1)$. By Young's inequality we have

$$\begin{aligned} |u| &= (p\alpha_n)^{1/p}|v| \cdot (p\alpha_n)^{-1/p}\|\nabla u\|_n \\ &\leqq \frac{1}{p}\{(p\alpha_n)^{1/p}|v|\}^p + \frac{1}{q}\{(p\alpha_n)^{-1/p}\|\nabla u\|_n\}^q \\ &= \alpha_n|v|^p + \frac{1}{\beta_n}\|\nabla u\|_n^n. \end{aligned} \tag{2.1}$$

Since $\int_\Omega |\nabla v|^n dx = 1$, it follows from the above theorem that

$$\int_\Omega \exp(\alpha_n|v|^p) dx \leqq c|\Omega|,$$

which together with (2.1) implies

$$\int_\Omega \exp|u| dx \leqq \exp\left(\frac{1}{\beta_n}\|\nabla u\|_n^n\right) \int_\Omega \exp(\alpha_n|v|^p) dx \leqq c|\Omega| \exp\left(\frac{1}{\beta_n}\|\nabla u\|_n^n\right).$$

The proof is complete.

Our aim of this section is a modification of the corollary to Moser's theorem of the Sobolev space $W^{1,p}(\Omega)$, where $W^{1,p}(\Omega)$ is the space of functions $u \in L^p(\Omega)$ whose weak derivative $\partial u/\partial x_i \in L^p(\Omega)$. First we extend this corollary to the case of radially symmetric functions.

Theorem 2.1. *Let $\Omega = \{x \in \boldsymbol{R}^n; |x| < L\} (n \geqq 2)$ and $u \in W^{1,n}(\Omega)$ with $u(x) = u(|x|)$. Then for any $\varepsilon > 0$ there exists a constant C_ε depending on ε and $|\Omega|$ such that*

$$\int_\Omega \exp|u| dx \leqq C_\varepsilon \exp\left\{ \left(\frac{1}{\beta_n} + \varepsilon \right) \|\nabla u\|_n^n + \frac{2^n}{n|\Omega|} \|u\|_1 \right\}. \tag{2.2}$$

Proof. Since $C^1(\bar{\Omega})$ is dense in $W^{1,n}(\Omega)$, it suffices to prove in the case $u \in C^1(\bar{\Omega})$. We can also assume $u(x) \geqq 0$. Put

$$v(x) = (u(x) - u(L))_+,$$

where $w_+ = \max\{w, 0\}$. Then

$$\nabla v = \begin{cases} \nabla u & \text{if } u(x) > u(L), \\ 0 & \text{otherwise.} \end{cases}$$

Since $v \in W_0^{1,n}(\Omega)$ and $\|\nabla v\|_n \leqq \|\nabla u\|_n$, by Corollary to the Trudinger-Moser Theorem we have

$$\int_\Omega \exp v \, dx \leqq c|\Omega| \exp\left(\frac{1}{\beta_n} \|\nabla u\|_n^n \right).$$

On the other hand,

$$\int_\Omega \exp v \, dx \geqq \exp\{-u(L)\} \int_\Omega \exp u \, dx.$$

Therefore,

$$\exp\{-u(L)\} \int_\Omega \exp u \, dx \leqq c|\Omega| \exp\left(\frac{1}{\beta_n} \|\nabla u\|_n^n \right),$$

and so

$$\int_\Omega \exp u \, dx \leqq c|\Omega| \exp\left(\frac{1}{\beta_n} \|\nabla u\|_n^n + u(L) \right). \tag{2.3}$$

In order to estimate $u(L)$, let us choose $r_0 \in [L/2, L)$ such that

$$u(r_0) r_0^{n-1} \leqq \frac{2}{L} \int_{L/2}^L u(r) r^{n-1} dr.$$

Then

$$u(r_0) \leqq r_0^{1-n} \frac{2}{L\omega_{n-1}} \int_\Omega u(x)dx \tag{2.4}$$

$$\leqq \left(\frac{L}{2}\right)^{1-n} \frac{2}{L\omega_{n-1}} \int_\Omega u(x)dx$$

$$= 2^n \frac{1}{L^n \omega_{n-1}} \int_\Omega u(x)dx = \frac{2^n}{n|\Omega|} \int_\Omega u(x)dx.$$

Here we used $|\Omega| = n^{-1}L^n\omega_{n-1}$. Since

$$u(L) = u(r_0) + \int_{r_0}^L u'(r)dr = u(r_0) + \int_{r_0}^L r^{(1-n)/n} u'(r) r^{(n-1)/n} dr,$$

by Hölder's inequality and (2.4) we have

$$u(L) \leqq u(r_0) + \left(\int_{r_0}^L r^{-1}dr\right)^{(n-1)/n} \left(\int_{r_0}^L |u'(r)|^n r^{n-1} dr\right)^{1/n}$$

$$\leqq u(r_0) + \left(\int_{L/2}^L r^{-1}dr\right)^{(n-1)/n} \left(\frac{1}{\omega_n}\right)^{1/n} \|\nabla u\|_n$$

$$\leqq u(r_0) + (\log 2)^{(n-1)/n} \left(\frac{1}{\omega_n}\right)^{1/n} \|\nabla u\|_n$$

$$\leqq \varepsilon \|\nabla u\|_n^n + \frac{2^n}{n|\Omega|} \int_\Omega u(x)dx + C_\varepsilon,$$

which together with (2.3) leads to (2.2). The proof is complete.

For the two-dimensional case without the restriction of the radially symmetric functions the following theorem is an immediate consequence of Proposition 2.3 by Chang and Yang [5].

Theorem ([5]). *Suppose Ω is a piecewise C^2, bounded, finitely connected domain in $\boldsymbol{R}^2$ with finite number of vertices. Let θ_Ω be the minimum interior angle at the vertices of Ω. Then there exists a constant c_Ω such that*

$$\int_\Omega \exp|u|dx \leqq C_\Omega \exp\left\{\frac{1}{8\theta_\Omega}\|\nabla u\|_2^2 + \frac{1}{|\Omega|}\|u\|_1\right\}$$

for $u \in W^{1,2}(\Omega)$.

We remark that $\theta_\Omega = \pi = \beta_2/16$ in the case where there are no corners on Ω.

Before stating Theorem 2.2 we prepare the Gagliardo-Nirenberg inequality.

Lemma (Gagliardo-Nirenberg inequality [10, p. 37]). *Let Ω be a bounded domain in $\boldsymbol{R}^n$ with C^m boundary, and let $u \in W^{m,q}(\Omega) \cap L^r(\Omega)$. Then the inequality*

$$\|u\|_{W^{k,p}} \leqq C\|u\|_{W^{m,q}}^{\theta} \cdot \|u\|_{L^r}^{1-\theta}$$

holds, if $k - n/p = \theta(m - n/q) - n(1-\theta)/r$, $1 \leqq p, q, r < \infty$, and $k/m \leqq \theta \leqq 1$. The constant C depends only on Ω, m, r, q, k.

For functions in $W^{1,n}(\Omega)$ on a n-dimensional domain Ω with smooth boundary we can show Theorem 2.2 by using a similar argument to that in Cherrier [6]. In order to prove we need a partition of unity subordinated to a finite covering $\{U(x_j)\}(x_j \in \partial\Omega)$ of the boundary $\partial\Omega$ to transform $u(x)$ on $U(x_j)$ onto $B_\rho^+ = \{\xi = (\xi_1, \dots, \xi_n) \in \boldsymbol{R}^n; |\xi| \leqq \rho, \xi_n > 0\}$ by a orthogonal transformation A, and then apply the Trudinger-Moser inequality to the extended function $v(\xi)(\in W_0^{1,n}(B_\rho))$ on B_ρ of the function $u(Ax)$, where $B_\rho = \{\xi \in \boldsymbol{R}^n; |\xi| \leqq \rho\}$. The proof also depends on the Gagliard-Nirenberg inequality.

Theorem 2.2. *Let Ω be a bounded domain in $\boldsymbol{R}^n (n \geqq 2)$ with smooth boundary. Then for any $\varepsilon > 0$ there exist positive constants C_ε and γ_ε such that*

$$\int_\Omega \exp|u| dx \leqq C_\varepsilon \exp\left\{\left(\frac{2}{\beta_n} + \varepsilon\right)\|\nabla u\|_n^n + \gamma_\varepsilon \|u\|_1^n\right\} \tag{2.5}$$

for $u \in W^{1,n}(\Omega)$.

3. Time global existence

This section is devoted to the existence of the global solution to (KS) in the case $\Omega \subset \boldsymbol{R}^2$ by making use of the Trudinger-Moser inequality shown in Section 2 and the theorem by Yagi [28]. To do so we need the boundedness of $\|u(\cdot, t)\|_2$ for $0 < t < T_{\max}$. In what follows we frequently denote

$$\|u(\cdot, t)\|_p = \left\{\int_\Omega |u(x,t)|^p \, dx\right\}^{1/p}$$

simply by $\|u\|_p$. First we remember the following

Theorem ([28]). *Let Ω be a bounded domain in $\boldsymbol{R}^2$. Assume u_0, $v_0 \in H^{1+\varepsilon_0}(\Omega)$ for some $0 < \varepsilon_0 \leqq 1$ and $u_0 \geqq 0$, $v_0 \geqq 0$ on Ω. Let $T_{\max}$ be the maximal existence time of (u, v).*

(i) (KS) *has a unique non-negative solution (u, v) satisfying*

$$u, v \in C([0, T_{\max}) : H^{1+\varepsilon_1}(\Omega)) \cap C^1((0, T_{\max}) : L^2(\Omega)) \cap C((0, T_{\max}) : H^2(\Omega))$$

for any $0 < \varepsilon_1 < \min\{\varepsilon_0, 1/2\}$. *Moreover* (u, v) *has further regularity properties:*

$$u \in C^1((0, T_{\max}) : H^1(\Omega)), \quad v \in C^{1/4}((0, T_{\max}) : L^3(\Omega)) \cap C^{5/4}((0, T_{\max}) : H^1(\Omega)).$$

(ii) *If* $T_{\max} < \infty$, *then*

$$\lim_{t \to T_{\max}} (\|u(\cdot, t)\|_{H^{1+\varepsilon_0}} + \|v(\cdot, t)\|_{H^{1+\varepsilon_0}}) = \infty,$$

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^p} = \infty \quad \textit{for any } 1 < p \leqq \infty,$$

$$\limsup_{t \to T_{\max}} (\|v(\cdot, t)\|_{H^{1+\varepsilon}} = \infty \quad \textit{for any } 0 < \varepsilon \leqq \varepsilon_0.$$

(iii) *There exists* $c > 0$ *such that if* $\int_\Omega u_0(x)dx < c$ *then the solution* (u, v) *of* (KS) *exists globally in time.*

In what follows by a solution of (KS) on $Q_T = \Omega \times (0, T)$ we mean a function (u, v) on Q_T such that

(i) $u, v \in C([0, T) : H^{1+\varepsilon_1}(\Omega)) \cap C^1((0, T) : L^2(\Omega)) \cap C((0, T) : H^2(\Omega))$.

(ii) (u, v) satisfies (KS) for $0 < t < T$.

Throughout the rest of this section and the next section we always assume that

$$u_0 \geqq 0, \quad u_0 \not\equiv 0 \quad \text{and } v_0 \geqq 0,$$

which assures the positivity of the solution (u, v) on $\bar{\Omega} \times (0, T_{\max})$ (see Lemma 3.1 below).

Let us regard $\gamma - \Delta$ as a closed operator in a Banach space. So define the closed operator A_p in $L^p(\Omega)(1 \leqq p < \infty)$ with domain $D(A_p)$ by

$$A_p = \gamma - \Delta, \quad D(A_p) = \left\{u \in W^{2,p}(\Omega); \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega\right\}.$$

The operator A_p is sectorial in $L^p(\Omega)$ and $\sigma(A) \subset \{z \in C; \mathscr{R}(z) > \gamma_0\}$ for a positive number γ_0, where $\sigma(A)$ is the spectrum of A_p. Then for $\beta \geqq 0$ the fractional powers A_p^β of A_p are defined, and the domain $X_p^\beta = D(A_p^\beta)$ is a Banach space under the norm $\|u\|_{X_p^\beta} = \|A_p^\beta u\|_p$. Since A_p is sectorial in $L^p(\Omega)$, the operator $-A_p$ generates the analytic semigroup $\{T_p(t)\}$. For fundamental properties of sectorial operators and analytic semigroups, we refer to [10, 22, 25]. As for X_p^β we know the following

Lemma ([10]). *Suppose* $\Omega \subset \boldsymbol{R}^n$ *a bounded domain with smooth boundary. Then for* $0 \leqq \beta \leqq 1$, *the following holds:*

$$X_p^\beta \subset W^{k,q}(\Omega) \qquad \text{when } k - n/q < 2\beta - n/p, \quad q \geqq p,$$

$$X_p^\beta \subset C^\nu(\Omega) \qquad \text{when } 0 \leqq \nu < 2\beta - n/p,$$

and the inclusion is continuous.

Lemma 3.1. *Let non-negative solution* (u, v) *to* (KS) *satisfies the following:*
(i) (u, v) *is a classical solution;*
(ii) $u(x, t) > 0,\ v(x, t) > 0$ *on* $\bar{\Omega} \times (0, T_{\max})$.

Proof. It suffices to show that (u, v) is a classical solution, because the positivity of (u, v) follows from the strong maximum principle for classical solutions to single parabolic equations.

For simplicity we put $\chi = \varepsilon = \gamma = \alpha = 1$. For fix $\tau \in (0, T_{\max}/2)$ the second equality of (KS) is rewritten as

$$v(t+\tau) = T_2(t)v(\tau) + \int_0^t T_2(t-s)u(s+\tau)ds \quad (0 < t < T_{\max} - \tau),$$

where $v(t) = v(\cdot, t)$ and $u(t) = (\cdot, t)$. For $\beta \in (0, 1/2)$ we have

$$A_2^\beta v(t+\tau) = T_2(t)A_2^\beta v(\tau) + \int_0^t T_2(t-s)A_2^\beta u(s+\tau)ds \quad (0 < t < T_{\max} - \tau).$$

By noting that $u \in C^1((0, T_{\max}) : H^1(\Omega))$ and $X_2^{1/2} = H^1(\Omega)$, there is a positive constant C_τ such that

$$\|A_2^\beta u(t+\tau) - A_2^\beta u(s+\tau)\|_2 \leqq C_\tau |t-s| \quad (t, s \in [0, T_{\max} - 2\tau]).$$

Then Lemma 3.5.1 in [10] yields that

$$A_2^\beta v(\cdot + \tau) \in C^1((0, T_{\max} - 2\tau) : X_2^\gamma) \quad \text{for any } \gamma \in [0, 1),$$

which implies that

$$v \in C^1((0, T_{\max}) : X_2^\beta) \quad \text{for any } \beta \in \left[0, \tfrac{3}{2}\right). \tag{3.1}$$

Next we rewrite the first equation of (KS) as

$$\frac{\partial u}{\partial t} = A_2 u + g,$$

where

$$g(t) = -\nabla v(t) \cdot \nabla u(t) - u(t)\Delta v(t) + u(t). \tag{3.2}$$

Direct calculations give us that for $t_1, t_2 \in (0, T_{\max})$

$$\|g(t_1) - g(t_2)\|_2 \leqq 2\|v(t_1) - v(t_2)\|_{X_2^1} \|u(t_1)\|_{X_2^{1/2}} + (2\|v(t_1)\|_{X_2^1} + 1)\|u(t_1) - u(t_2)\|_{X_2^{1/2}},$$

from which together with (3.1) it follows that for fix $\tau \in (0, T_{\max})$ there is a positive constant C_τ such that

$$\|g(t_1) - g(t_2)\|_2 \leqq C_\tau |t_1 - t_2| \quad \text{for } t_1, t_2 \in [\tau, T_{\max} - \tau].$$

Hence, by Theorem 3.5.2 in [10] we have

$$u \in C^1((0, T_{\max}) : X_2^\beta) \quad \text{for any } \beta \in [0, 1), \tag{3.3}$$

which implies

$$\frac{\partial u}{\partial t} \in C^\nu(\bar{\Omega} \times (0, T_{\max})) \quad \text{for any } \nu \in [0, 1). \tag{3.4}$$

By using a similar way to that used above, it is also obtained that

$$v \in C^1((0, T_{\max}) : X_2^\beta) \quad \text{for any } \beta \in [0, 2),$$

which implies that

$$v, \frac{\partial u}{\partial t}, \Delta v \in C^\nu(\bar{\Omega} \times (0, T_{\max})) \quad \text{for any } \nu \in [0, 1), \tag{3.5}$$

because of $X_2^\beta \subset C^\nu(\Omega)$ for $0 \leqq \nu < 2\beta - 1$ and $0 \leqq \beta < 2$.

For fix $t \in (0, T_{\max})$ let us rewrite the first equation of (KS) as

$$\begin{cases} \Delta u - u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$h = \frac{\partial u}{\partial t} + g$$

and g is the same as (3.2). Noting that

$$X_2^\beta \subset W^{1,q}(\Omega) \quad \text{for} \quad 2 \leqq q < \frac{1}{1-\beta}, \quad 0 \leqq \beta < 1,$$

by (3.3), (3.4) and (3.5) we see that

$$h \in C((0, T_{\max}) : L^q(\Omega)) \quad \text{for any } q \in [2, \infty).$$

Hence,

$$u \in C((0, T_{\max}) : W^{2,q}(\Omega)) \quad \text{for any } q \in [2, \infty).$$

which together with $W^{2,q}(\Omega) \subset C^\nu(\Omega)(0 \leqq \nu < 2 - 2/q)$ yields that

$$\frac{\partial u}{\partial x_i} \in C((0, T_{\max}) : C^\nu(\Omega)) \quad \text{for any } \nu \in (0, 1) \ (i = 1, 2).$$

By this, $h \in C((0, T_{\max}) : C^{\nu}(\Omega))$ for any $\nu \in (0,1)$. Hence, we have

$$u \in C((0, T_{\max}) : C^{2+\nu}(\Omega)) \quad \text{for any } \nu \in (0,1).$$

Therefore, (u, v) is a classical solution of (KS).

Lemma 3.2. *The following holds:*

$$\|u(\cdot, t)\|_1 = \|u_0\|_1,$$

and

$$\|v(\cdot, t)\|_1 = e^{-(\gamma/\varepsilon)t}\|v_0\|_1 + \frac{\alpha}{\gamma}\|u_0\|_1(1 - e^{-(\gamma/\varepsilon)t}).$$

Proof. We integrate the first equation of (KS) and use Green's formula to obtain

$$\frac{d}{dt}\int_\Omega u(x,t)dx = 0,$$

from which it follows that

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx. \tag{3.6}$$

Next by the second equation of (KS) and Green's formula we have

$$\frac{d}{dt}\int_\Omega vdx = -\frac{\gamma}{\varepsilon}\int_\Omega vdx + \frac{\alpha}{\varepsilon}\int_\Omega udx.$$

Put $V(t) = \int_\Omega vdx$. Then by (3.6) we have

$$\frac{d}{dt}V(t) = -\frac{\gamma}{\varepsilon}V(t) + \frac{\alpha}{\varepsilon}\|u_0\|_1,$$

from which it follows that

$$V(t) = e^{-(\gamma/\varepsilon)t}V(0) + \frac{\alpha}{\gamma}\|u_0\|_1(1 - e^{-(\gamma/\varepsilon)t}).$$

The proof is complete.

Lemma. 3.3. *Put*

$$W(t) = \int_\Omega\left\{u\log u - \chi uv + \frac{\chi}{2\alpha}(|\nabla v|^2 + \gamma v^2)\right\}dx.$$

Then we have

$$\frac{dW}{dt}(t) + \frac{\chi\varepsilon}{\alpha}\int_\Omega (v_t)^2dx + \int_\Omega u|\nabla\cdot(\log u - \chi v)|^2dx = 0.$$

Proof. Multiplying $\log u - \chi v$ by the first equation of (KS) and using Green's formula, we have

$$\int_\Omega u_t(\log u - \chi v)dx = \int_\Omega \nabla \cdot (\nabla u - \chi u \nabla v)(\log u - \chi v)dx \tag{3.7}$$
$$= -\int_\Omega u|\nabla \cdot (\log u - \chi v)|^2 dx.$$

Noting that $\int_\Omega u_t\, dx = 0$ holds, we have

$$\int_\Omega u_t(\log u - \chi v)dx = \int_\Omega (u \log u)_t\, dx - \int_\Omega u_t\, dx - \chi \int_\Omega u_t v dx \tag{3.8}$$
$$= \frac{d}{dt}\int_\Omega u \log u dx - \chi \frac{d}{dt}\int_\Omega uv dx + \chi \int_\Omega u v_t\, dx.$$

Since, from the second equation of (KS),

$$u = \frac{\varepsilon}{\alpha} v_t - \frac{1}{\alpha}\Delta v + \frac{\gamma}{\alpha} v,$$

we have

$$\int_\Omega u v_t\, dx = \frac{\varepsilon}{\alpha}\int_\Omega (v_t)^2 dx - \frac{1}{\alpha}\int_\Omega (\Delta v) v_t\, dx + \frac{\gamma}{\alpha}\int_\Omega v v_t\, dx.$$

From Green's formula it follows that

$$\int_\Omega (\Delta v) v_t\, dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx,$$

and so

$$\int_\Omega u v_t\, dx = \frac{\varepsilon}{\alpha}\int_\Omega (v_t)^2 dx + \frac{1}{2\alpha}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \frac{\gamma}{2\alpha}\frac{d}{dt}\int_\Omega v^2 dx. \tag{3.9}$$

By (3.8) and (3.9) we have

$$\int_\Omega u_t(\log u - \chi v)dx = \frac{d}{dt}\int_\Omega \left\{u \log u - \chi uv + \frac{\chi}{2\alpha}(|\nabla v|^2 + \gamma v^2)\right\}dx + \frac{\chi\varepsilon}{\alpha}\int_\Omega (v_t)^2 dx,$$

which together with (3.7) leads to

$$\frac{dW}{dt}(t) + \frac{\chi\varepsilon}{\alpha}\int_\Omega (v_t)^2 dx = -\int_\Omega u|\nabla \cdot (\log u - \chi v)|^2 dx.$$

The proof is complete.

We proceed to the estimation of $\int_\Omega uv dx$ and $\int_\Omega u \log u dx$. To do so, we use the Trudinger-Moser inequality derived in Section 2 and an analogous argument in the proof of Theorem 2 of Biler and Nadzieja [2] to show the following

Lemma 3.4. *Let $W(t)$ be the same as in Lemma 3.3 and let*

$$\pi^* = \begin{cases} 8\pi & \text{if } \Omega = \{x \in \mathbf{R}^2; |x| < L\} \quad \text{and } (u_0, v_0) \quad \text{is radial in } x, \\ 4\pi & \text{otherwise.} \end{cases}$$

If $\int_\Omega u_0\, dx < \pi^*/(\alpha\chi)$, there exists a positive constant C independent of t such that

$$\int_\Omega uv dx \leqq C \quad \text{and } |W(t)| \leqq C.$$

Proof. For any $\delta > 0$ we rewrite $W(t)$ as

$$\begin{aligned} W(t) &= \int_\Omega \{u \log u - (\chi + \delta)uv\} dx + \int_\Omega \left\{\delta uv + \frac{\chi}{2\alpha}(|\nabla v|^2 + \gamma v^2)\right\} dx \\ &= -\int_\Omega u \log \frac{e^{(\chi+\delta)v}}{u}\, dx + \int_\Omega \left\{\delta uv + \frac{\chi}{2\alpha}(|\nabla v|^2 + \gamma v^2)\right\} dx. \end{aligned} \tag{3.10}$$

Put

$$M = \int_\Omega u_0(x) dx.$$

Then from Lemma 3.2, we have

$$M = \int_\Omega u(x,t) dx.$$

Since $-\log x$ is a convex function and

$$\int_\Omega \frac{u}{M}\, dx = 1,$$

it follows from Jensen's inequality that

$$\begin{aligned} -\log\left\{\frac{1}{M}\int_\Omega e^{(\chi+\delta)v(x,t)} dx\right\} &= -\log \int_\Omega \frac{e^{(\chi+\delta)v(x,t)}}{u}\frac{u}{M}\, dx \\ &\leqq \int_\Omega \left(-\log \frac{e^{(\chi+\delta)v(x,t)}}{u}\right)\frac{u}{M}\, dx \\ &= -\frac{1}{M}\int_\Omega u \log \frac{e^{(\chi+\delta)v(x,t)}}{u}\, dx. \end{aligned} \tag{3.11}$$

By the Trudinger-Moser inequality for any $\varepsilon > 0$ there exists a constant $C > 0$

such that

$$\int_\Omega e^{(\chi+\delta)v}\,dx \leqq C\exp\left\{\left(\frac{1}{2\pi^*}+\varepsilon\right)(\chi+\delta)^2\|\nabla v\|_2^2+\frac{2(\chi+\delta)}{|\Omega|}\|v\|_1\right\},$$

that is,

$$\log\left\{\frac{1}{M}\int_\Omega e^{(\chi+\delta)v}\,dx\right\} \leqq \log\frac{C}{M}+\left(\frac{1}{2\pi^*}+\varepsilon\right)(\chi+\delta)^2\|\nabla v\|_2^2+\frac{2(\chi+\delta)}{|\Omega|}\|v\|_1.$$

From this together with (3.10) and (3.11) we have

$$\begin{aligned} W(t) &\geqq -M\log\left\{\frac{1}{M}\int_\Omega e^{(\chi+\delta)v(x,t)}\,dx\right\}+\int_\Omega\left\{\delta uv+\frac{\chi}{2\alpha}(|\nabla v|^2+\gamma v^2)\right\}dx \\ &\geqq -M\left\{\log\frac{C}{M}+\left(\frac{1}{2\pi^*}+\varepsilon\right)(\chi+\delta)^2\|\nabla v\|_2^2+\frac{2(\chi+\delta)}{|\Omega|}\|v\|_1\right\} \\ &\quad+\int_\Omega\left\{\delta uv+\frac{\chi}{2\alpha}(|\nabla v|^2+\gamma v^2)\right\}dx. \end{aligned}$$

Consequently we have

$$\begin{aligned} &\left\{\frac{\chi}{2\alpha}-M\left(\frac{1}{2\pi^*}+\varepsilon\right)(\chi+\delta)^2\right\}\|\nabla v\|_2^2+\delta\int_\Omega uv\,dx \\ &\qquad\leqq M\left\{\log\frac{C}{M}+\frac{2(\chi+\delta)}{|\Omega|}\|v\|_1\right\}+W(t)\leqq C+W(0). \end{aligned}$$

If we can choose $\delta>0$ and $\varepsilon>0$ such that

$$\left\{\frac{\chi}{2\alpha}-M\left(\frac{1}{2\pi^*}+\varepsilon\right)(\chi+\delta)^2\right\}>0, \tag{3.12}$$

our assertion holds. This is possible, because the condition of this lemma implies $\alpha\chi M/\pi^*<1$, from which we can take $\delta>0$ such that

$$\frac{\alpha\chi M}{\pi^*}\left(\frac{\chi+\delta}{\chi}\right)^2<1.$$

Then we choose ε such that

$$\alpha\chi M\left(\frac{1}{\pi^*}+2\varepsilon\right)\left(\frac{\chi+\delta}{\chi}\right)^2<1,$$

from which (3.12) follows. The proof is complete.

Remark 3.1. From Lemmas 3.3 and 3.4 we see that there exists a constant

$C > 0$ such that

$$\int_0^t \|v_t(\cdot, s)\|_2^2 ds \leqq C. \tag{3.13}$$

In fact, Lemma 3.3 we have

$$-\int_\Omega u|\nabla \log u - \chi v|^2 dx = \frac{d}{dt} W(t) + \frac{\chi\varepsilon}{\alpha}\int_\Omega (v_t)^2 dx,$$

which leads to, by integration to the both sides,

$$\int_0^t \|v_t(\cdot, t)\|_2^2 dt \leqq W(0) - W(t) \leqq |W(0)| + |W(0)| \leqq C.$$

Thus we have our assertion.

The following lemma is a modification of the inequality ([3])

$$\|f\|_3^3 \leqq \varepsilon\|f\|_{H^1}^2 \|f \log|f|\|_1 + C_\varepsilon \|f\|_1$$

for any $f \in H^1(\Omega)$, where Ω is a domain $\boldsymbol{R}^2$ with smooth boundary. The proof is done by using a similar way to that in [3] and the following inequality

$$\|f\|_p \leqq C\|f\|_{H^1}^{1-1/p} \cdot \|f\|_1^{1/p} \quad (2 \leqq p < \infty)$$

by the Gagliardo-Nirenberg inequality.

Lemma 3.5. *Let Ω be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary, and let $2 \leqq p < \infty$. Then for any $\varepsilon > 0$ there exist positive constants k_1 and k_2 depending on ε such that $k_1 \to 0$ and $k_2 \to \infty$ as $\varepsilon \to 0$, and*

$$\|f\|_p \leqq \varepsilon\|\nabla f\|_2^{1-1/p} \|f \log|f|\|_1^{1/p} + k_1\|f \log|f|\|_1 + k_2\|f\|_1^{1/p}$$

for any $f \in H^1(\Omega)$.

Now we show the following

Lemma 3.6. *There exists a positive constant C such that $\|u(\cdot, t)\|_2 \leqq C$.*

Proof. In this proof the coefficients of the system (KS) are not essential. To avoid the complication of notations we put $\chi = \varepsilon = \gamma = \alpha = 1$ in (KS). Multiply u by the first equation of (KS). Then we have

$$\int_\Omega u_t u dx = \int_\Omega \nabla u u dx - \int_\Omega \nabla \cdot (u \nabla v) u dx,$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx = -\int_{\Omega} \nabla \cdot (u\nabla v) u dx. \tag{3.14}$$

As for the right hand side, we have

$$\begin{aligned} -\int_{\Omega} \nabla \cdot (u\nabla v) u dx &= \int_{\Omega} (u\nabla u) \cdot \nabla v dx \\ &= \frac{1}{2}\int_{\Omega} \nabla u^2 \cdot \nabla v dx = -\frac{1}{2}\int_{\Omega} u^2 \nabla v dx \\ &= -\frac{1}{2}\int_{\Omega} u^2 (v_t + v - u) dx \\ &\leqq -\frac{1}{2}\int_{\Omega} u^2 v_t dx + \frac{1}{2}\int_{\Omega} u^3 dx. \end{aligned} \tag{3.15}$$

It follows from Lemma 3.5 that

$$\|u\|_3 \leqq \varepsilon \|\nabla u\|_2^{2/3} \|u \log u\|_1^{1/3} + C(\|u \log u\|_1 + \|u\|_1^{1/3}).$$

By Hörder's inequality and the Gagliardo-Nirenberg inequality we have

$$\begin{aligned} \int_{\Omega} |u^2 v_t| dx &\leqq \|v_t\|_2 \|u\|_4^2 \\ &\leqq C\|v_t\|_2((\|\nabla u\|_2^{1/2} + \|u\|_2^{1/2})\|u\|_2^{1/2})^2 \\ &\leqq C\|v_t\|_2(\|\nabla u\|_2 \|u\|_2 + \|u\|_2^2) \\ &\leqq \varepsilon\|\nabla u\|_2^2 + C(\|v_t\|_2^2 + \|v_t\|_2)\|u\|_2^2, \end{aligned}$$

which together with (3.14) and (3.15) leads to

$$\begin{aligned} \frac{d}{dt}\|u\|_2^2 + 2\|\nabla u\|_2^2 \leqq \varepsilon\|\nabla u\|_2^2 &+ C(\|v_t\|_2^2 + \|v_t\|_2)\|u\|_2^2 \\ &+ \varepsilon^3 \|\nabla u\|_2^2 \|u \log u\|_1 + C(\|u \log u\|_1^3 + \|u\|_1). \end{aligned}$$

that is,

$$\begin{aligned} &\frac{d}{dt}\|u\|_2^2 + (2 - \varepsilon - \varepsilon^3 \|u \log u\|_1)\|\nabla u\|_2^2 \\ &\qquad \leqq C\{(\|v_t\|_2^2 + \|v_t\|_2)\|u\|_2^2 + \|u \log u\|_1^3 + \|u\|_1\}. \end{aligned} \tag{3.16}$$

Since $\|u \log u\|_1$ is bounded, we can choose $\varepsilon > 0$ such that $2 - \varepsilon -$

$\varepsilon^3\|u \log u\|_1 \geqq 1$. Thus it follows from (3.16) that

$$\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 \leqq C\{(\|v_t\|_2^2 + \|v_t\|_2)\|u\|_2^2 + \|u \log u\|_1^3 + \|u\|_1\}. \tag{3.17}$$

By the Gagliardo-Nirenberg inequality we have

$$\|u\|_2 \leqq C(\|\nabla u\|_2^{1/2}\|u\|_1^{1/2} + \|u\|_1),$$

that is,

$$\begin{aligned}\|u\|_2^2 &\leqq C(\|\nabla u\|_2\|u\|_1 + \|u\|_1^2)\\ &\leqq \|\nabla u\|_2^2 + C\|u\|_1^2,\end{aligned}$$

from which it follows that

$$\|\nabla u\|_2^2 \geqq \|u\|_2^2 - C\|u\|_1^2. \tag{3.18}$$

Thus from (3.17) and (3.18) we have

$$\frac{d}{dt}\|u\|_2^2 + \|u\|_2^2 \leqq C\{(\|v_t\|_2^2 + \|v_t\|_2)\|u\|_2^2 + L\}, \tag{3.19}$$

where

$$L = C \sup_{t>0}\{\|u \log u\|_1^3 + \|u\|_1 + \|u\|_1^2\}.$$

Putting $h(t) = \|v_t(\cdot, t)\|_2^2$ and $f(t) = \|u(\cdot, t)\|_2^2$, we have, by (3.19),

$$f'(t) + f(t) \leqq C\left(h(t) + \sqrt{h(t)}\right)f(t) + L.$$

Since $C\sqrt{h(t)} \leqq 1/2 + C^2 h(t)/2$, we have

$$f'(t) + \left(\frac{1}{2} - c \cdot h(t)\right)f(t) \leqq L, \tag{3.20}$$

where $c = C + C^2/2$. Define $\varphi(t)$ by

$$\varphi(t) = \int_0^t \left(\frac{1}{2} - c \cdot h(s)\right)ds.$$

Then it follows from (3.20) that

$$f(t) \leqq f(0)e^{-\varphi(t)} + Le^{-\phi(t)}\int_0^t e^{\varphi(s)}ds. \tag{3.21}$$

Remembering (3.13) in Remark 3.1, we have

$$-\varphi(t) = -\frac{1}{2}t + c\int_0^t h(s)ds \leqq -\frac{1}{2}t + C$$

with some positive constant C. Taking this into account, we have, from (3.21)

$$f(t) \leqq C(f(0)e^{-(1/2)t} + L),$$

which implies

$$\|u(\cdot, t)\|_2 \leqq C$$

with some positive constant C which is independent of t. The proof is complete.

Now since we proved the boundedness of $\|u(\cdot, t)\|_2$ for $0 \leqq t < T_{\max}$, by Yagi's theorem we see $T_{\max} = \infty$, which means the time global existence of the solution of (KS).

4. L^∞ estimate

In this section we prove the second property

$$\sup_{t \geqq 0}\{\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty\} < \infty \tag{4.1}$$

in Theorem 1.1 by making use of Moser's technique (see Alikakos [1]). In what follows we assume as before that Ω is a bounded domain in $\boldsymbol{R}^2$, and we put $\chi = \alpha = 1$ in (KS) for simplicity.

Lemma 4.1. *Let* $2/3 < \beta < 1$. *Then there exists constants* $C > 0$ *and* λ_2 *such that for any* $t > 0$

$$\|\nabla v(\cdot, t)\|_3 \leqq C\left\{\frac{e^{-\lambda_2 t}}{t^\beta}\|v_0\|_2 + M_1\int_0^t \frac{e^{-\lambda_2 s}}{s^\beta}\,ds\right\},$$

where $M_1 = \sup_{t>0}\|u(t)\|_2$.

Proof. The second equality of (KS) is rewritten as

$$v(t) = T_2(t)v_0 + \int_0^t T_2(t-s)u(s)ds \quad (t > 0).$$

Consider

$$A_2^\beta v(t) = A_2^\beta T_2(t)v_0 + A_2^\beta\int_0^t T_2(t-s)u(s)\,ds$$

$$= A_2^\beta T_2(t)v_0 + \int_0^t A_2^\beta T_2(t-s)u(s)\,ds,$$

from which there exist constants $\lambda_2 > 0$ and $C > 0$ such that

$$\|A_2^\beta v(t)\|_2 \leqq C\left\{\frac{e^{-\lambda_2 t}}{t^\beta}\|v_0\|_2 + \int_0^t \frac{e^{-\lambda_2(t-s)}}{(t-s)^\beta}\|u(s)\|_2\,ds\right\}$$
$$\leqq C\left\{\frac{e^{-\lambda_2 t}}{t^\beta}\|v_0\|_2 + M_1\int_0^t \frac{e^{-\lambda_2 s}}{s^\beta}\,ds\right\}.$$

From the assumption on β the inclusion $D(A_2^\beta) \subset W^{1,3}(\Omega)$ is continuous. Thus there exists a constant $C > 0$ such that

$$\|\nabla v(t)\|_3 \leqq C\|A_2^\beta v(t)\|_2$$
$$\leqq C\left\{\frac{e^{-\lambda_2 t}}{t^\beta}\|v_0\|_2 + M_1\int_0^t \frac{e^{-\lambda_2 s}}{s^\beta}\,ds\right\},$$

which completes the proof.

Lemma 4.2. *For any $\tau > 0$ there exists a constant $C_\tau > 0$ such that*

$$\|v(\cdot,t)\|_\infty + \|\nabla v(\cdot,t)\|_\infty \leqq C_\tau \quad \textit{for } t \geqq \tau. \tag{4.2}$$

Proof. From Lemmas 3.6 and 4.1 for any $\tau > 0$ there exists a positive constant $C_\tau > 0$ depending on $\|v_0\|_2, \sup_{t>0}\|u(\cdot,t)\|_2$ and τ such that

$$\|\nabla v\|_3 \leqq C_\tau \quad \text{for } t \geqq \tau/2, \tag{4.3}$$

where $\|\nabla v\|_3 = \|\nabla v(\cdot,t)\|_3$. Then we proceed to the proof of the boundedness of $\|u(\cdot,t)\|_4$. Multiply the first equation of (KS) by u^3 to get

$$\frac{1}{4}\frac{d}{dt}\int_\Omega w^2 dx + \frac{3}{4}\int_\Omega |\nabla w|^2 dx = -\int_\Omega u^3\nabla\cdot(u\nabla v)dx = \int_\Omega \nabla(u^3)\cdot u\nabla v dx \tag{4.4}$$
$$= 3\int_\Omega u^3\nabla u\cdot\nabla v dx = \frac{3}{2}\int_\Omega u^2\nabla(u^2)\cdot\nabla v dx$$
$$= \frac{3}{2}\int_\Omega w\nabla w\cdot\nabla v dx,$$

where $w(x,t) = u^2(x,t)$. As for the final part of the equality above, by Hölder's inequality and the Gagliardo-Nirenberg inequality we have

$$\left|\int_\Omega w\nabla w\cdot\nabla v dx\right| \leqq \frac{1}{3}\|\nabla w\|_2^2 + C\|\nabla v\|_3^{12}\|w\|_1^2 \tag{4.5}$$
$$\leqq \frac{1}{3}\|\nabla w\|_2^2 + M \quad \text{for } t \geqq \tau/2,$$

where $M = \sup_{t\geqq\tau/2} C\|\nabla v\|_3^{12}\|u\|_2^2$. Here we note that it follows from (4.3) and

Lemma 3.6 that $M < \infty$. By (4.4) and (4.5) we have

$$\frac{d}{dt}\int_{\Omega} w^2 dx + \int_{\Omega} |\nabla w|^2 dx \leqq 6M \quad \text{for } t \geqq \tau/2. \tag{4.6}$$

By the Gagliardo-Nirenberg inequality and the Schwarz inequality there exists a $\delta > 0$ such that

$$\|w\|_2^2 \leqq \frac{1}{\delta}(\|\nabla w\|_1^2 + \|w\|_1^2),$$

which together with (4.6) leads to

$$\frac{d}{dt}\int_{\Omega} w^2 dx + \delta \int_{\Omega} w^2 dx \leqq 6M + \|u\|_2^2 \leqq 6M + L \quad \text{for } t \geqq \tau/2, \tag{4.7}$$

where $L = \sup_{t>0}\|u(\cdot, t)\|_2^2$. It follows from (4.7) that

$$\|w(\cdot, t)\|_2^2 \leqq \|u(\cdot, \tau/2)\|_4^4 e^{-\delta(t-\tau/2)} + \frac{1}{\delta}(6M + L) \quad \text{for } t \geqq \tau/2,$$

which means that there exists a constant $C_\tau > 0$ depending on τ, $\|v_0\|_2$ and $\sup_{t>0}\|u\|_2$ such that

$$\|u(\cdot, t)\|_4 \leqq C_\tau \quad \text{for } t \geqq \tau/2.$$

Finally we show (4.2). Let $3/4 < \beta < 1$, and consider

$$A_4^\beta v(t) = A_4^\beta T_4(t - \tau/2)v(\tau/2) + \int_{\tau/2}^{t} A_4^\beta T_4(t - s)u(s)ds.$$

Then there exists constants $C > 0$ and $\lambda_4 > 0$ such that

$$\begin{aligned} \|A_4^\beta v(t)\|_4 &\leqq C\left\{\frac{e^{-\lambda_4(t-\tau/2)}}{(t-\tau/2)^\beta}\|v(\tau/2)\|_4 + \int_{\tau/2}^{t} \frac{e^{-\lambda_4(t-s)}}{(t-s)^\beta}\|u(s)\|_4 ds\right\} \\ &\leqq C\left\{\frac{e^{-\lambda_4(t-\tau/2)}}{(t-\tau/2)^\beta}\|v(\tau/2)\|_4 + M_4 \int_{\tau/2}^{t} \frac{e^{-\lambda_4(t-s)}}{(t-s)^\beta} ds\right\}, \end{aligned} \tag{4.8}$$

where $M_4 = \sup_{t>\tau/2}\|u(t)\|_4$. From (4.8) it follows that

$$\|A_4^\beta v(t)\|_4 \leqq C_\tau \quad \text{for } t \geqq \tau. \tag{4.9}$$

Since $3/4 < \beta < 1$, the inclusion $D(A_4^\beta) \subset C^1(\bar{\Omega})$ is continuous. Thus from (4.9) we have

$$\|v(\cdot, t)\|_\infty + \|\nabla v(\cdot, t)\|_\infty \leqq C'\|A_4^\beta v(t)\|_4 \leqq C_\tau \quad \text{for } t \geqq \tau,$$

which completes the proof.

We are now in a position to prove the boundedness of (u, v).

Proof of (4.1). For any $\tau > 0$ there exists a constant $C_\tau > 0$ such that

$$\|u(\cdot, t)\|_\infty \leqq C_\tau \max\{1, \|u(\cdot, \tau)\|_1, \|u(\cdot, \tau)\|_\infty\} \quad \text{for } t \geqq \tau. \tag{4.10}$$

In fact let $1 \leqq p < \infty$ and multiply u^p by the first equation in (KS). From Green's formula we have

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega u^{p+1} &= -p\int_\Omega u^{p-1}|\nabla u|^2 dx + p\int_\Omega u^p \nabla u \cdot \nabla v dx \\ &= -\frac{4p}{(p+1)^2}\int_\Omega |\nabla u^{(p+1)/2}|^2 dx + p\int_\Omega u^p \nabla u \cdot \nabla v dx.\end{aligned}$$

From Lemma 4.2 and the Schwarz inequality it follows

$$\begin{aligned}p\int_\Omega u^p \nabla u \cdot \nabla v dx &\leqq pC_\tau\left(\int_\Omega u^{p+1}\, dx\right)^{1/2}\left(\int_\Omega u^{p-1}|\nabla u|^2 dx\right)^{1/2} \\ &\leqq \frac{2p}{(p+1)^2}\int_\Omega |\nabla u^{(p+1)/2}|^2 dx + \frac{p}{2}C_\tau\int_\Omega u^{p+1}\, dx \quad \text{for } t \geqq \tau.\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\int_\Omega u^{p+1}\, dx \leqq -\frac{2p}{p+1}\int_\Omega |\nabla u^{(p+1)/2}|^2 dx + \frac{p(p+1)}{2}C_\tau\int_\Omega u^{p+1}\, dx,$$

which yields (4.10) by using Moser's technique (see Alikakos [1]). On the other hand since

$$u, v \in C([0, \infty) : H^{1+\varepsilon_1}(\Omega)),$$

and

$$H^{1+\varepsilon_1}(\Omega) \subset C(\bar{\Omega}),$$

there exists a constant $K_\tau > 0$ such that

$$\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \leqq K_\tau \qquad \text{for } 0 \leqq t \leqq \tau. \tag{4.11}$$

From (4.2), (4.10) and (4.11) it follows that

$$\sup_{t \geqq 0}\{\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty\} < \infty.$$

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References

[1] Alikakos, N. D., L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations, **4** (1979), 827–868.

[2] Biler, P. and Nadzieja, T., Existence and nonexistence of solutions for a model of gravitational interactions of particles, I, Colloq. Math., **66** (1994), 319–334.

[3] Biler, P., Hebisch, W. and Nadzieja, T., The Debye system: Existence and large time behavior of solutions, Nonlinear Anal., **238** (1994), 1189–1209.

[4] Biler, P., Existence and nonexistence of solutions for a model of gravitational interactions of particles, III, Colloq. Math., **68** (1995), 229–239.

[5] Chang, S. Y. A. and Yang, P. C., Conformal deformation of metrics on S^2, J. Differential Geom., **27** (1988), 259–296.

[6] Cherrier, P., Meilleures constantes dans des inégalités relatives aux espaces de Sobolev, Bull. Sc. Math., **108** (1984), 225–262.

[7] Childress, S., Chemotactic collapse in two dimensions, Lecture Notes in Biomath., **55**, Springer, Berlin-Heidelberg-New York, 1984, 61–66.

[8] Childress, S. and Percus, J. K., Nonlinear aspects of chemotaxis, Math. Biosci., **56** (1981), 217–237.

[9] Diaz, J. I. and Nagai, T., Symmetrization in a parabolic-elliptic system related to chemotaxis, Adv. Math. Sci. Appl., **5** (1995), 659–680.

[10] Henry, D., Geometric Theory of Semilinear Parabolic Equations, Lecture Note in Math., **840**, Springer, 1981.

[11] Herrero, M. A. and Velázquez, J. J. L., Singularity patterns in a chemotaxis model, Math. Ann., to appear.

[12] Herrero, M. A. and Velázquez, J. J. L., Chemotaxis collapse for the Keller-Segel model, J. Math. Biol., to appear.

[13] Herrero, M. A. and Velázquez, J. J. L., A blow up mechanism for a chemotaxis model, preprint.

[14] Jäger, W. and Luckhaus, S., On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc., **329** (1992), 819–824.

[15] Keller, E. F. and Segel, L. A., Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26**, (1970), 399–415.

[16] Lions, J. L. and Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications*, Springer, Berlin-Heidelberg-New York, 1972.

[17] Mizutani, Y. and Nagai, T., Self-similar radial solutions to a system of partial differential equations modelling chemotaxis, Bull. Kyushu Inst. Tech. (Math. Natur, Sci), **42** (1995), 19–28.

[18] Moser, J., A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J., **20** (1971), 1077–1092.

[19] Nagai, T., Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., **5** (1995), 581–601.

[20] Nagai, T. and Senba, T., Global existence and blow-up of radial solutions to a parabolicelliptic system of chemotaxis, submitted.

[21] Nanjundiah, V., Chemotaxis, signal relaying, and aggregation morphology, J. Theor. Biol., **42** (1973), 63–105.

[22] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York-Berlin-Heidelberg-Tokyo, 1983.

[23] Rascle, M. and Ziti, C., Finite time blow-up in some models of chemotaxis, J. Math. Biol., **33** (1995), 388–414.

[24] Senba, T., Blow-up of radially symmetric solutions to some systems of partial differential equations modelling chemotaxis, Adv. Math. Sci. Appl., to appear.

[25] Tanabe, H., *Equations of Evolution*, Pitman, London, 1979.

[26] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[27] Trudinger, N. S., On imbeddings into Orlicz spaces and some applications, J. Math. Mech., **17** (1967), 473–483.

[28] Yagi, A., Norm behavior of solutions to the parabolic system of chemotaxis, Math. Japonica, to appear.

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