Funkcialaj Ekvacioj, **40** (1997) 293–312

Global Solutions to the Initial-Boundary Value Problem for the Quasilinear Visco-Elastic Wave Equation with a Perturbation

By

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1. Introduction

In this paper we shall investigate the global existence and decay of solutions to the initial-boundary value problem for the quasilinear wave equation with a viscosity and a nonlinear perturbation:

$$\text{(P)}\qquad \begin{cases} u_{tt} - \operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\} - \Delta u_t + g(u) = 0 & \text{in } [0,\infty)\times\Omega, \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x) \ \text{and} \ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$. As nonlinearities we want to treat, for example, $\sigma(v^2) = 1/\sqrt{1+v^2}$ and $g(u) = -|u|^\alpha u$.

The problem (P) with $N = 1$ and $g(u) \equiv 0$ was proposed by Greenberg [10] and Greenberg, MacCamy & Mizel [11] as a model of quasilinear wave equation which admits a global solution for large data. Physically, this represents a longitudinal motion of a visco-elastic material. (For more physical background see [23].) Since then, related problems have been investigated by many authors from various points of view ([1]–[6], [8], [13]–[25], [27]–[30] etc.). The term $g(u) = -|u|^\alpha u$, $\alpha > 0$, is often called as "blowing up" term, and existence or nonexistence of global solutions to the wave equation with such nonlinear term has been discussed by many authors (Cf. [9], [12], [15]–[16], [24], [28] etc.).

When $\sigma(v^2) = |v|^p$, $p \geq 0$, and $|g(u)| \leq k_0|u|^{\alpha+1}$, $0 < \alpha < (p(N+2)+4)/(N-p-2)^+$, the global existence of solutions was discussed in Nakao & Nanbu [21] by use of a "stable set" method. But, the method in [21] cannot be applied to the case $\sigma(v^2) = 1/\sqrt{1+v^2}$.

Quite recently, Kobayashi, Pecher & Shibata [14] has treated such nonlinearity and proved the global existence of smooth solutions to the problem (P) with $g(u) \equiv 0$. Subsequently, one of the present authors [19] has derived a decay estimate of the solutions under the assumption that the mean curvature of $\partial\Omega$ is nonpositive.

The object of this paper is to combine the method in [19] with a concept of stable set to prove the global existence of the problem (P) with σ and g like

$\sigma = 1/\sqrt{1+v^2}$ and $g(u) = -|u|^{\alpha}u$, $\alpha > 0$, respectively. The main difficulty in our problem lies in the fact that the operator $-\mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\}$ is not coercive in a usual sense, that is,

$$\lim_{\|\nabla u\|\to\infty, u\in H_1^0} \frac{\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 dx}{\|\nabla u\|} = \infty$$

does not hold. Caused by this non-coercivity, it is difficult to control the nonlinear perturbation $g(u)$.

To overcome this difficulty we treat the so-called H_2-solutions instead of usual energy finite solutions and derive a precise decay estimate for the energy.

We note that our restriction on the growth order α is weaker than the usual one. Indeed, when $\sigma(v^2) \equiv 1$, we can take $0 < \alpha < 4/(N-4)^+$, which is much weaker than the usual restriction $0 < \alpha < 4/(N-2)^+$.

Finally we note that our method can be applied to another typical nonlinearity $\sigma(v^2) = |v|^p, p > 0$, and yields a new existence theorem of global solutions. We discuss on this topic at the last section.

2. Statement of the result

We make the following assumptions on the nonlinear terms $\sigma(v^2)$ and $g(u)$.

Hyp.A. $\sigma(\cdot)$ is a differentiable function on $\boldsymbol{R}^+ \equiv [0,\infty)$ and satisfies the conditions

$$k_0\sigma(v^2)v^2 \le \int_0^{v^2} \sigma(\eta)d\eta \le k_1\sigma(v^2)v^2 \tag{2.1}$$ (i)

and

$$k_0(1+v^2)^{-\beta} \le \sigma(v^2) \tag{2.2}$$

with some k_0, $k_1 > 0$ and

$$0 < \beta < \frac{2}{N-2} \qquad (0 < \beta < \infty \text{ if } N = 1,2). \tag{2.3}$$

$$\sigma(v^2) + 2\sigma'(v^2)v^2 \ge 0 \quad \text{and} \quad \sup_{v\in\boldsymbol{R}}\{\sigma(v^2) + |\sigma'(v^2)|v^2\} < \infty. \tag{2.4}$$ (ii)

Hyp.B. $g(\cdot)$ is a differentiable function on $\boldsymbol{R}$ and satisfies

$$|g(u)| \le k_1|u|^{\alpha+1} \tag{2.5}$$

and

$$|g'(u)| \le k_1|u|^\alpha \tag{2.6}$$

with

$$0 < \alpha < \frac{4-2\beta(N-2)}{(N-4)^+} \qquad (0 < \alpha < \infty \text{ if } 1 \le N \le 4). \tag{2.7}$$

Remark. (1) If $\sigma = 1/\sqrt{1+v^2}$, we can take $\beta = 1/2$. Hence, (2.3) is valid for $1 \le N \le 5$. (2) When $\beta = 0$, we can apply a usual stable set method. For this, however, we must require, at least, $H_1^0 \subset L^{\alpha+2}$, i.e., $0 < \alpha \le 4/(N-2)$. Our condition (2.7) is much weaker than this, which comes from considering H_2-solutions and utilizing a decay estimate. (3) We could replace (2.5) and (2.6) by weaker ones

$$|g(u)| \le k_1(|u|^{\alpha_1+1} + |u|^{\alpha_2+1}) \tag{2.5$'$}$$

and

$$|g'(u)| \le k_1(|u|^{\alpha_1} + |u|^{\alpha_2}) \tag{2.6$'$}$$

with $0 < \alpha_1 < \alpha_2$, α_2 satisfying (2.7).

Hyp.C. $\partial\Omega$ is C^2-class and the mean curvature of $\partial\Omega$ is nonpositive with respect to the outward normal.

Our result reads as follows.

Theorem 1. *Under the* Hyp.A, B *and* C, *there exists an open* (*unbounded*) *set* $\mathscr{S} \subset H_1^0 \cap H_2 \times H_1^0$ *including* $(0,0)$ *such that if* $(u_0, u_1) \in \mathscr{S}$, *the problem* (P) *admits a unique solution* $u(t)$ *in the class*

$$L^\infty([0,\infty); H_1^0 \cap H_2) \cap W^{1,2}([0,\infty); H_1^0) \cap W^{1,\infty}([0,\infty); L^2),$$

satisfying the decay estimate

$$E_0(t) \le \begin{cases} C_1 e^{-\lambda t} & \text{for some } \lambda > 0 \text{ if } N = 1, \\ C_1(L)(1+t)^{-L} & \text{for any } L \gg 1 \text{ if } N = 2, \\ C_1(1+t)^{-(1+2/\beta(N-2))} & \text{if } N \ge 3, \end{cases}$$

where C_1 *denotes constants depending on* $\|u_0\|_{H^2} + \|u_1\|_{H^1}$ *and we set*

$$E_0(t) \equiv \|u_t(t)\|^2 + \int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 dx$$

($\|\cdot\|$ *denotes* L^2-*norm on* Ω).

Remark. If we assume further $g(u)u \geq 0$ and $0 < \alpha < 4/(N-2)^+$, then we can prove the global existence of an energy finite solution for all $(u_0, u_1) \in H_1^0 \cap H_2 \times H_1^0$ (Cf. [27]).

Theorem 2 (Existence of strong solution). *In addition to the assumptions in Theorem* 1 *we let* $u_1 \in H_1^0 \times H_2$. *Then, the solution* $u(t)$ *in Theorem* 1 *belongs to*

$$C^2([0,\infty); L^2) \cap C^1([0,\infty); H_1^0 \cap H_2) \cap W^{2,2}([0,\infty); H_1^0),$$

that is, $u(t)$ *is a strong solution in* L^2 *sense.*

The precise definition of $\mathscr{S}$, the set of initial data, will be given in the course of the proof.

3. A stable set

Here, we introduce a certain set in $H_1^0 \cap H_2$, where the solutions are expected to stay for all time if the initial data are small. For convenience we call it as "stable set".

Let us introduce some functionals defined on $H_1^0 \cap H_2$. We set

$$F(\nabla u) = \frac{1}{2}\int_\Omega \int_0^{|\nabla u|^2} \sigma(\eta)d\eta dx, \tag{3.1}$$

$$\tilde{F}(\nabla u) = \int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 dx, \tag{3.2}$$

$$J(u) = F(\nabla u) + \int_\Omega G(u)dx \qquad \left(G(u) \equiv \int_0^u g(\eta)d\eta\right) \tag{3.3}$$

and

$$\tilde{J}(u) = \tilde{F}(\nabla u) + \int_\Omega g(u)udx. \tag{3.4}$$

The following lemma is proved in [19].

Lemma 3.1. *Let* $0 < \beta \leq N/(N-2)$ $(0 < \beta < \infty$ *if* $N = 1,2)$ *and set*

$$\varepsilon = \frac{N - \beta(N-2)}{N + \beta(N-2)} \text{ if } N \geq 3, \qquad \varepsilon = \text{arbitrary close to } 1 \text{ if } N = 2$$

and $\varepsilon = 1$ *if* $N = 1$. *Then, we have*

$$\|\nabla u\|_{1+\varepsilon}^2 \leq CF(\nabla u)(1 + \|\Delta u\|^{2\beta}) \tag{3.5}$$

for $u \in H_1^0 \cap H_2$, *where* $C = C(\varepsilon)$ *is a constant.*

Proof. For completeness we sketch the proof briefly. We consider the case $N \geq 3$. Then,

$$\begin{aligned}\int_\Omega |\nabla u|^{1+\varepsilon} dx &= \int_\Omega \frac{|\nabla u|^{1+\varepsilon}}{(1+|\nabla u|^2)^{\beta(1+\varepsilon)/2}}(1+|\nabla u|^2)^{\beta(1+\varepsilon)/2} dx \\ &\leq C\left(\int_\Omega \frac{|\nabla u|^2}{(1+|\nabla u|^2)^\beta} dx\right)^{(1+\varepsilon)/2} (1+\|\nabla u\|_{2\beta(1+\varepsilon)/(1-\varepsilon)}^{\beta(1+\varepsilon)}) \\ &\leq CF(\nabla u)^{(1+\varepsilon)/2}(1+\|\Delta u\|^{\beta(1+\varepsilon)}),\end{aligned}$$

where we have used Hyp.A, (i) and the fact that $2\beta(1+\varepsilon)/(1-\varepsilon) \leq 2N/(N-2)$.

Q.E.D.

Lemma 3.2. *For any $K > 0$, there exists $\varepsilon_0 \equiv \varepsilon_0(K) > 0$ such that if $\|\Delta u\| \leq K$ and $\|\nabla u\|_{1+\varepsilon} \leq \varepsilon_0$, then*

$$J(u) \geq C_0(K)\|\nabla u\|_{1+\varepsilon}^2 \quad \text{and} \quad \tilde{J}(u) \geq C_0(K)\|\nabla u\|_{1+\varepsilon}^2 \tag{3.6}$$

for some constant $C_0(K) > 0$.

Proof. Note that $W^{1,1+\varepsilon} \subset L^{\alpha+2}$ if $1/(\alpha+2) \geq 1/(1+\varepsilon) - 1/N$. In this case we see

$$\|u\|_{\alpha+2}^{\alpha+2} \leq C\|\nabla u\|_{1+\varepsilon}^{\alpha+2}. \tag{3.7}$$

In other case: $1/(\alpha+2) < 1/(1+\varepsilon) - 1/N$, we see by Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|u\|_{\alpha+2}^{\alpha+2} &\leq C\|u\|_{N(1+\varepsilon)/(N-1-\varepsilon)}^{(\alpha+2)(1-\theta)}\|\Delta u\|^{(\alpha+2)\theta} \\ &\leq C\|\nabla u\|_{1+\varepsilon)}^{(\alpha+2)(1-\theta)}\|\Delta u\|^{(\alpha+2)\theta}\end{aligned} \tag{3.8}$$

with

$$\theta = \left(\frac{1}{1+\varepsilon} - \frac{1}{N} - \frac{1}{\alpha+2}\right)\left(\frac{2}{N} + \frac{1}{1+\varepsilon} - \frac{1}{N} - \frac{1}{2}\right)^{-1}. \tag{3.9}$$

Here, we note that

$$\begin{aligned}(\alpha+2)(1-\theta) &= \left(\frac{2(\alpha+2)}{N} - \frac{\alpha}{2}\right)\left(\frac{1}{N} + \frac{\beta(N-2)}{2N}\right)^{-1} \\ &= \frac{4(\alpha+2) - N\alpha}{2+\beta(N-2)} > 2,\end{aligned} \tag{3.10}$$

where we have used the assumption (2.7) on α. We set $\theta = 0$ if $1/(\alpha+2) \geq$

$1/(1+\varepsilon)-1/N$. Then, (3.8) holds in any case and we have, by Lemma 3.1,

$$\begin{aligned} J(u) &\geq F(\nabla u) - k_1\|u\|_{\alpha+2}^{\alpha+2} \\ &\geq \tilde{C}\|\nabla u\|_{1+\varepsilon}^2(1+\|\Delta u\|^{2\beta})^{-1} - C\|\nabla u\|_{1+\varepsilon}^{(\alpha+2)(1-\theta)}\|\Delta u\|^{(\alpha+2)\theta} \\ &\geq \{\tilde{C}(1+K^{2\beta})^{-1} - C\|\nabla u\|_{1+\varepsilon}^{(\alpha+2)(1-\theta)-2}K^{(\alpha+2)\theta}\}\|\nabla u\|_{1+\varepsilon}^2. \end{aligned} \tag{3.11}$$

Thus, we can use (3.10) to define $\varepsilon_0 \equiv \varepsilon_0(K)$ by

$$C\varepsilon_0^{(\alpha+2)(1-\theta)-2}K^{(\alpha+2)\theta} = \frac{1}{2}\tilde{C}(1+K^{2\beta})^{-1} \equiv C_0(K) \tag{3.12}$$

and to get

$$J(u) \geq C_0(K)\|\nabla u\|_{1+\varepsilon}^2 \tag{3.13}$$

if $\|\nabla u\|_{1+\varepsilon} \leq \varepsilon_0$. It is clear that (3.13) remains valid for $\tilde{J}(u)$. Q.E.D.

Let us define our stable set $\mathscr{W}_K$, $K>0$ in the following way:

$$\mathscr{W}_K \equiv \{u \in H_1^0 \cap H_2 \mid \|\Delta u\| < K \text{ and } \|\nabla u\|_{1+\varepsilon} < \varepsilon_0\}.$$

Remark. If we assume $g(u)u \geq 0$, then we need not take $\varepsilon_0(K)$, and $\mathscr{W}_K$ is replaced by $\tilde{\mathscr{W}}_K = \{u \in H_1^0 \cap H_2 \mid \|\Delta u\| < K\}$.

4. A priori estimates

We shall derive some a priori estimates for an assumed strong solution $u(t)$ as in Theorem 2. The proof of Theorems will follow from these estimates combined with standard compactness arguments.

Proposition 4.1. *Let $u(t)$ be a strong solution satisfying $u(t) \in \mathscr{W}_K$ on $[0,T)$ for some $K>0$. Then, it holds that*

$$E(t) \equiv \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) \leq g_K(t) \qquad \text{on } [0,T), \tag{4.1}$$

where $g_K(t)$ is defined by

$$g_K(t) = \begin{cases} C(E(0))e^{-\lambda t} & \text{if } N=1, \\ (E(0)^{-1/\nu} + C_1(K)^{-1}(t-1)^+)^{-\nu} & \text{if } N\geq 2 \end{cases} \tag{4.2}$$

with $\nu = 1+2/\beta(N-2)$ (ν is arbitrary large if $N=2$). Here, $C_1(K)$ is a constant such that $C_1(K) \sim CK^{2\theta_0+2\beta(1-\theta_0)}$ as $K\to\infty$.

Proof. Under the assumption $u(t) \in \mathscr{W}_K$, $J(u(t))$ and $\tilde{J}(u(t))$ are both equivalent to $F(\nabla u)$ by Lemma 3.2. Thus, the estimate (4.2) follows from the argument in [19]. For convenience of the readers, however, we sketch it briefly. We may assume $T > 1$.

Multiplying the equation by u_t and integrating we have

$$\int_t^{t+1} \|\nabla u_t(s)\|^2 ds = E(t) - E(t+1) \equiv D(t)^2 \tag{4.3}$$

and

$$\int_0^T \|\nabla u_t(s)\|^2 ds \le E(0) < \infty. \tag{4.4}$$

From (4.3), there exist $t_1 \in [t, t+1/4]$, $t_2 \in [t+3/4, t+1]$ such that

$$\|\nabla u_t(t_i)\| \le 2D(t), \qquad i = 1, 2.$$

Next, multiplying the equation by u and integrating we see

$$\begin{aligned} \int_{t_1}^{t_2} \tilde{J}(u(s))ds = &-(u_t(t_2), u(t_2)) + (u_t(t_1), u(t_1)) \\ &+ \int_{t_1}^{t_2} \|u_t(s)\|^2 ds - \int_{t_1}^{t_2} (\nabla u_t(s), \nabla u(s))ds. \end{aligned} \tag{4.5}$$

Here,

$$\begin{aligned} \int_{t_1}^{t_2} |(\nabla u_t(s), \nabla u(s))| ds &\le \left(\int_{t_1}^{t_2} \|\nabla u_t(s)\|^2 ds \right)^{1/2} \sup_{t \le s \le t+1} \|\nabla u(s)\| \\ &\le CD(t) \sup_{t \le s \le t+1} \|\nabla u(s)\|_{1+\varepsilon}^{1-\theta_0} \|\Delta u(s)\|^{\theta_0} \\ &\le \tilde{C}_1(K) D(t) E(t)^{(1-\theta_0)/2} \end{aligned} \tag{4.6}$$

with

$$\tilde{C}_1(K) = CC_0(K)^{-(1-\theta_0)/2} K^{\theta_0} = CK^{\theta_0}(1 + K^{2\beta})^{(1-\theta_0)/2} \tag{4.7}$$

and

$$\theta_0 = \frac{\beta(N-2)}{\beta(N-2)+2} \tag{4.8}$$

($\theta_0 = 0$ if $N = 1$ and arbitrary close to 0 if $N = 2$), where we have used the fact $J(u) \ge (1/2)F(\nabla u)$ by the way of the choice of $\varepsilon_0(K)$ (see (3.12)).

Similarly, we have

$$\begin{aligned} |(u_t(t_i), u(t_i))| &\le C\|\nabla u_t(t_i)\| \|\nabla u(t_i)\| \\ &\le \tilde{C}_1(K) D(t) E(t)^{(1-\theta_0)/2}. \end{aligned} \tag{4.9}$$

It follows from (4.4)–(4.6) and (4.9) that

$$E(t_2) \leq \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} E(s)ds$$
$$\leq CD(t)^2 + \tilde{C}_1(K)D(t)E(t)^{(1-\theta_0)/2}.$$

Hence,

$$E(t) \equiv E(t_2) + \int_t^{t_2} \|\nabla u_t(s)\|^2 ds$$
$$\leq CD(t)^2 + \tilde{C}_1(K)D(t)E(t)^{(1-\theta_0)/2} \qquad (\tilde{C}_1(K) \equiv C\tilde{C}_1(K))$$

or

$$E(t)^{1+\theta_0} \leq \tilde{C}_1^2(K)D(t)^2 \equiv C_1(K)(E(t) - E(t+1)),$$

which implies (4.2) (note that $1/\theta_0 = \nu$). Q.E.D.

The following estimate is the heart of this paper.

Proposition 4.2. *Let $u(t)$ be a strong solution satisfying $u(t) \in \mathscr{W}_K$ on $[0, T)$ for some $K > 0$. Then, we have*

$$\|\Delta u(t)\|^2 \leq I_0^2 + q(K, E(0)) \equiv Q^2(K, I_0, E(0)) \tag{4.10}$$

for all $t \in [0, T)$, where $q(K, E(0))$ is a certain quantity such that $\lim_{E(0)\to 0} q(K, E(0)) = 0$ and

$$I_0^2 \equiv C(\|\Delta u_0\|^2 + |(\Delta u_0, u_1)| + E(0)).$$

Proof. Multiplying the equation by $-\Delta u$ we have

$$\frac{d}{dt}\left\{\frac{1}{2}\|\Delta u(t)\|^2 + (\nabla u_t(t), \nabla u(t))\right\} + \int_\Omega \mathrm{div}(\sigma\nabla u)\Delta u dx \tag{4.11}$$
$$= \|\nabla u_t(t)\|^2 + \int_\Omega g(u)\Delta u dx.$$

Here, by the Hyp.C we see ([19], [5])

$$\int_\Omega \mathrm{div}(\sigma\cdot\nabla u)\Delta u dx \tag{4.12}$$
$$= \int_\Omega (\sigma + 2\sigma'|\nabla u|^2)|D^2 u|^2 dx - (N-1)\int_{\partial\Omega} \sigma H(x) d\Gamma \geq 0,$$

where $H(x)$ is the mean curvature of $\partial\Omega$ at $x \in \partial\Omega$. Thus, we have from (4.11) and (4.4) that

$$(4.13)\qquad \frac{1}{2}\|\Delta u(t)\|^2-(\Delta u(t),u_t(t))\le\frac{1}{2}\|\Delta u_0\|^2+|(\Delta u_0,u_1)|+E(0)$$
$$+k_1\int_0^t\int_\Omega |u(s)|^\alpha|\nabla u(s)|^2dxds.$$

We shall show that the last term in the right-hand side of (4.13) is bounded. We consider two cases $0<\alpha\le 4/(N-2)$ and $4/(N-2)<\alpha$ separately. Of course, we assume $\alpha<(4-2\beta(N-2))/(N-4)^+$ in any case.

The case: $0<\alpha\le 4/(N-2)$ ($0<\alpha<\infty$ if $N=1,2$).

In this case,

$$(4.14)\qquad \int_\Omega |u|^\alpha|\nabla u|^2dx$$
$$\le\left(\int_\Omega |u|^{2N/(N-2)}dx\right)^{(N-2)\alpha/2N}\left(\int_\Omega |\nabla u|^{4N/(2N-(N-2)\alpha)}dx\right)^{1-(N-2)\alpha/2N}$$
$$\le C\|\nabla u\|^\alpha\|\nabla u\|^2_{4N/(2N-(N-2)\alpha)}$$
$$\le C\|\nabla u\|_{1+\varepsilon}^{\alpha(1-\theta_1)}\|\Delta u\|^{\alpha\theta_1}\|\nabla u\|_{1+\varepsilon}^{2(1-\theta_2)}\|\Delta u\|^{2\theta_2}$$
$$\le C(C_0^{-1}(K)g_K(t))^{(\alpha(1-\theta_1)/2+1-\theta_2}K^{\alpha\theta_1+2\theta_2}\equiv C_2(K)g_K(t)^{\alpha(1-\theta_1)/2+1-\theta_2}$$

with

$$C_2(K)=C(1+K^{2\beta})^{\alpha(1-\theta_1)/2+1-\theta_2}K^{\alpha\theta_1+2\theta_2},$$

where

$$(4.15)\qquad \theta_1=\left(\frac{1}{1+\varepsilon}-\frac{1}{2}\right)\left(\frac{1}{N}+\frac{1}{1+\varepsilon}-\frac{1}{2}\right)^{-1}=\frac{(1-\varepsilon)N}{2N-(N-2)(1+\varepsilon)}$$

and

$$(4.16)\qquad \theta_2=\left(\frac{1}{1+\varepsilon}-\frac{2N-(N-2)\alpha}{4N}\right)\left(\frac{1}{N}+\frac{1}{1+\varepsilon}-\frac{1}{2}\right)^{-1}$$
$$=\frac{4N-(2N-(N-2)\alpha)(1+\varepsilon)}{2(2N-(N-2)(1+\varepsilon))}.$$

We note that

$$(4.17)\qquad \gamma\equiv\left\{\frac{\alpha(1-\theta_1)}{2}+(1-\theta_2)\right\}\nu$$
$$=\frac{(4-N)\alpha+4}{2\beta(N-2)}>1.$$

Therefore, recalling the definition of $g_K(t)$, we see

$$(4.18)\quad \int_0^t\int_\Omega |u(s)|^\alpha|\nabla u(s)|^2dxds \le C_2(K)\int_0^\infty \{E(0)^{-1/\nu}+C_1^{-1}(K)(t-1)^+\}^{-\gamma}dt$$
$$= C_2(K)\left(E(0)^{\gamma/\nu}+\frac{C_1(K)}{\gamma-1}E(0)^{(\gamma-1)/\nu}\right)$$
$$\equiv q(K,E(0)).$$

(4.10) follows immediately follows from (4.13) and (4.18). The case $N=1,2$ follows more easily by modifying the above argument.

The case: $4/(N-2)<\alpha<(4-2\beta(N-2))/(N-4)^+$ $(4/(N-2)<\alpha<\infty$ if $N=3,4)$.

In this case we see

$$(4.19)\quad \int_\Omega |u|^\alpha|\nabla u|^2dx \le \|u\|_{N\alpha/2}^\alpha\|\nabla u\|_{2N/(N-2)}^2$$
$$\le C\|\nabla u\|_{1+\varepsilon}^{(1-\theta_3)\alpha}\|\Delta u\|^{\theta_3\alpha}\|\Delta u\|^2$$

with

$$(4.20)\quad \theta_3=\left(\frac{1}{1+\varepsilon}-\frac{1}{N}-\frac{2}{N\alpha}\right)\left(\frac{2}{N}+\frac{1}{1+\varepsilon}-\frac{1}{N}-\frac{1}{2}\right)^{-1}$$
$$=\frac{2}{\alpha}\cdot\frac{N\alpha-(\alpha+2)(1+\varepsilon)}{N+2-(N-2)\varepsilon}\ (<1).$$

Hence

$$(4.21)\quad \int_\Omega |u|^\alpha|\nabla u|^2dx \le CK^{2+\theta_3\alpha}\|\nabla u(t)\|_{1+\varepsilon}^{(1-\theta_3)\alpha}$$
$$\le CC_3(K)g_K(t)^{(1-\theta_3)\alpha/2}$$

with

$$C_3(K)=C(1+K^{2\beta})^{(1-\theta_3)\alpha/2}K^{2+\theta_3\alpha}.$$

We again note that

$$(4.22)\quad \gamma\equiv\frac{(1-\theta_3)\alpha\nu}{2}=\frac{(4-N)\alpha+4}{2\beta(N-2)}>1.$$

Therefore, we obtain as in (4.18)

$$(4.23)\quad \int_0^t\int_\Omega |u(s)|^\alpha|\nabla u(s)|^2dxds \le q(K,E(0))$$

with $C_2(K)$ replaced by $C_3(K)$ in the definition of $q(E(0), K)$. Thus, (4.10) is proved also for the case $\alpha > 4/(N-2)$. Q.E.D.

5. Proof of theorems

On the basis of the inequality (3.11) and the estimate (4.10) we define

$$S_K = \left\{(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0 | Q(K, I_0, E(0)) < K \text{ and } \sqrt{C_0(K)^{-1}E(0)} < \varepsilon_0\right\}$$

and set

$$\mathscr{S} = \bigcup_{K>0} S_K.$$

By (3.11) and (4.10) we conclude that if $u(t)$ is a strong solution with $(u_0, u_1) \in S_K$, then $u(t) \in \mathscr{W}_K$ for all $t \geq 0$ and all the estimates in the previous section are valid for $u(t)$ on $[0, \infty)$.

For the proof of Theorems we employ Galerkin method. Let $\{w_j\}_{j=1}^{\infty}$ be the basis of H_1^0 consisted by the eigenfunction of $-\Delta$ with the Dirichlet condition. We define as is usual, the approximate solutions $u_m(t) \equiv \sum_{j=1}^{m} \lambda_j^m(t) w_j$, $m = 1, 2, 3, \ldots$, by

$$(5.1)\quad (\ddot{u}_m(t), w_j) + (\sigma(|\nabla u_m(t)|^2)\nabla u_m(t), \nabla w_j) - (\Delta \dot{u}_m(t), w_j) + (g(u_m(t)), w_j) = 0,$$

where $u_m(0)$ and $\dot{u}_m(0)$ should be determined in such a way that

$$u_m(0) \to u_0 \text{ in } H_2 \cap H_1^0 \qquad \text{and} \qquad \dot{u}_m(0) \to u_1 \text{ in } H_1^0$$

as $m \to \infty$.

By the theory of ordinary differential equations (5.1) has the unique solution $u_m(t)$. Suppose that $(u_0, u_1) \in S_K$ for some $K > 0$. Then, $(u_m(0), \dot{u}_m(0)) \in S_K$ for large m. It is clear that the argument in the previous section can be applied to $u_m(t)$ and, in particular, $u_m(t)$ exists on $[0, \infty)$. Since all the assertions in section 4 remain to hold for $u_m(t)$, we conclude that $u_m(t) \in \mathscr{W}_K$ for all $t > 0$, and all the estimates are valid for $u_m(t)$ in $[0, \infty)$.

Thus, $u_m(t)$ converges along a subsequence to $u(t)$ in the following way:

$$u_m(t) \to u(t) \qquad \text{weakly* in } L^\infty_{\text{loc}}([0, \infty); H_1^0 \cap H_2),$$
$$\dot{u}_m(t) \to u_t(t) \qquad \text{weakly* in } L^2_{\text{loc}}([0, \infty); H_1^0) \cap L^\infty_{\text{loc}}([0, \infty); L^2).$$

Since

$$L^\infty_{\text{loc}}([0, \infty); H_1^0 \cap H_2) \cap W^{1,2}_{\text{loc}}([0, \infty); H_1^0)$$

is compactly imbedded in $L^2_{\mathrm{loc}}([0,\infty);H^0_1)$, we know further

$$u_m(t)\to u(t)\qquad \text{strongly in } L^2_{\mathrm{loc}}([0,\infty);H^0_1) \text{ and a.e. in } [0,\infty)\times\Omega,$$

and hence,

$$\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\to\sigma(|\nabla u(t)|^2)\nabla u(t)\qquad \text{in } L^2_{\mathrm{loc}}([0,\infty);L^2).$$

Thus, the limit function $u(t)$ is a required solution belonging to

$$L^\infty([0,\infty);H^0_1\cap H_2)\cap W^{1,2}([0,\infty);H^0_1)\cap W^{1,\infty}([0,\infty);L^2).$$

Uniqueness follows from a similar argument as in [19]. Indeed, letting $u(t)$ and $v(t)$ be two solutions and setting $U=u_t$, $V=v_t$ and $W=U-V$, we see

$$W_t-\Delta W=\operatorname{div}\{\sigma(|\nabla u|^2)\nabla u-\sigma(|\nabla v|^2)\nabla v\}-g(u)+g(v). \tag{5.2}$$

Multiplying (5.2) by W and integrating we have

$$\begin{aligned}(5.3)\ \|W(t)\|^2&+2\int_0^t\|\nabla W(s)\|^2ds\\ &=-2\int_0^t\int_\Omega\{\sigma(|\nabla u|^2)\nabla u-\sigma(|\nabla v|^2)\nabla v\}\nabla W dxds-2\int_0^t\int_\Omega(g(u)-g(v))Wdxds\\ &\le C(K)\int_0^t\|\nabla(u-v)\|\|\nabla W(s)\|ds+C\int_0^t\int_\Omega(|u|^\alpha+|v|^\alpha)|u-v||W(s)|dxds\\ &\le C(K)t\int_0^t\|\nabla W(s)\|^2ds+C(K)\int_0^t\|\nabla(u-v)\|\|\nabla W(s)\|^2ds\\ &\le C(K)t\int_0^t\|\nabla W(s)\|^2ds.\end{aligned}$$

Hence, there exists $T_0>0$ such that

$$W(t)=0\qquad \text{on } [0,T_0].$$

Repeating this argument we obtain $W(t)=0$ on $[0,\infty)$, i.e., $u(t)=v(t)$ on $[0,\infty)$. The proof of Theorem 1 is now completed.

For the proof of Theorem 2 we shall derive further a priori estimates. Differentiating the equation we have

$$u_{ttt}-\operatorname{div}\{\sigma(|\nabla u|^2)\nabla u_t+2\sigma'(|\nabla u|^2)(\nabla u_t\cdot\nabla u)\nabla u\}-\Delta u_{tt}=g'(u)u_t. \tag{5.4}$$

Then, multiplying the equation (5.4) by u_{tt} we have

$$\begin{aligned}(5.5)\qquad &\frac{1}{2}\frac{d}{dt}\|u_{tt}(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \\ &\le \int_\Omega (\sigma + 2|\sigma'||\nabla u|^2)|\nabla u_t||\nabla u_{tt}|dx + C\int_\Omega |u|^\alpha |u_t||u_{tt}|dx \\ &\le C\|\nabla u_t(t)\|\|\nabla u_{tt}(t)\| + C\left(\int_\Omega |u|^{2\alpha}|u_t|^2 dx\right)^{1/2}\|u_{tt}(t)\|.\end{aligned}$$

Here, by a similar argument as in (4.14) and (4.18) we can prove

$$(5.6)\qquad \int_\Omega |u|^{2\alpha}|u_t|^2 dx \le C\|u(t)\|_{H_2}^{2\alpha}\|\nabla u_t(t)\|^2 \le C(K)\|\nabla u_t(t)\|^2.$$

Thus, we obtain

$$(5.7)\qquad \frac{d}{dt}\|u_{tt}(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \le C(K)\|\nabla u_t(t)\|^2,$$

which implies

$$\begin{aligned}(5.8)\qquad \|u_{tt}(t)\|^2 + \int_0^\infty \|\nabla u_{tt}(s)\|^2 ds &\le \|u_{tt}(0)\|^2 + C(K)\int_0^\infty \|\nabla u_t(s)\|^2 ds \\ &\le C(\|\Delta u_0\|, \|\Delta u_1\|) < \infty.\end{aligned}$$

Further, returning to the equation we have easily

$$\begin{aligned}(5.9)\qquad \|\Delta u_t(t)\| &\le \|u_{tt}(t)\| + \|\mathrm{div}\{\sigma \cdot \nabla u\}\| + \|g(u)\| \\ &\le C(\|\Delta u_0\|, \|\Delta u_1\|) < \infty.\end{aligned}$$

It is clear that the estimates (5.8) and (5.9) can be applied to the approximate solution $u_m(t)$. Thus, we see

$$\dot{u}_m(t) \to u_t(t) \qquad \text{weakly* in } L^\infty_{\mathrm{loc}}([0,\infty); H_1^0 \cap H_2)$$

and

$$\ddot{u}_m(t) \to u_{tt}(t) \qquad \text{weakly* in } L^\infty_{\mathrm{loc}}([0,\infty); L^2) \cap L^2_{\mathrm{loc}}([0,\infty); H_1^0).$$

Hence, the solution $u(t)$ in Theorem 1 further satisfies

$$(5.10)\qquad u_{tt} \in L^\infty([0,\infty); L^2) \cap L^2([0,\infty); H_1^0)$$

and

$$u_t \in L^\infty([0,\infty); H_1^0 \cap H_2) \cap L^2([0,\infty); H_1^0), \tag{5.11}$$

and the estimates (5.8) and (5.9) remain to hold for $u(t)$.

To prove that $u_{tt} \in C([0,\infty); L^2)$ and $u_t \in C([0,\infty); H_1^0 \cap H_2)$ we first note (Cf. [20]) that

$$u_{tt} \in L^\infty([0,\infty); L^2) \cap L^2([0,\infty); H_1^0) \subset C_w([0,\infty); L^2).$$

Then, by a standard argument (see Strauss [26]) we can prove

$$\begin{aligned} &\|u_{tt}(t)\|^2 - \|u_{tt}(s)\|^2 \\ &\quad + 2\int_s^t \int_\Omega \{\sigma(|\nabla u|^2)\nabla u_t + 2\sigma'(|\nabla u|^2)(\nabla u_t \cdot \nabla u)\nabla u\} \cdot \nabla u_{tt} dx d\tau \\ &\quad + 2\int_s^t \|\nabla u_{tt}(\tau)\|^2 d\tau \\ &= \int_s^t \int_\Omega g'(u_t) u_t u_{tt} dx d\tau \end{aligned} \tag{5.12}$$

for all $t \geq s \geq 0$.

The identity (5.12) easily implies that $\|u_{tt}(t)\|$ is continuous in t, and we conclude that $u_{tt} \in C([0,\infty); L^2)$. Finally, returning to the equation we easily see

$$\Delta u_t = u_{tt} - \operatorname{div}\{\sigma \nabla u\} + g(u) \in C([0,\infty); L^2).$$

The proof of Theorem 2 is completed.

Remark. If we assume $g(u)u \geq 0$, then S_K can be replaced by

$$\tilde{S}_K = \{(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0 | Q(K, I_0, E(0)) < K\}.$$

In particular, if $g(u)u \geq 0$ and $\lim_{K\to\infty} Q(K, I_0, E(0))/K = 0$, then we see $\mathscr{S} = H_2 \cap H_1^0 \times H_1^0$.

6. Another typical equation

In this section we consider the problem (P) with $\sigma(v^2)$ like $\sigma = |v|^p$, $p \geq 0$. More precisely, we make the following assumptions instead of Hyp.A and Hyp.B.

Hyp.A′. $\sigma(\cdot)$ is continuous on $\boldsymbol{R}^+$, differentiable on $(0,\infty)$ and satisfies the following conditions.

(6.1) $$k_0 \sigma(v^2) v^2 \leq \int_0^{v^2} \sigma(\eta) d\eta \leq k_1 \sigma(v^2) v^2 \qquad \text{(i)}$$

and

$$k_0|v|^p \le \sigma(v^2) \le k_1(|v|^q + 1) \tag{6.2}$$

with some k_0, $k_1 > 0$, where we assume

$$0 \le p \le q < \frac{4}{(N-2)^+}. \tag{6.3}$$

$$\sigma(v^2) + 2\sigma'(v^2)v^2 \ge 0, u \neq 0 \text{ and } \lim_{v\to 0}\sigma'(v^2)v^2 = 0. \tag{6.4}$$ (ii)

Hyp.B′. $g(\cdot)$ is a differentiable function on $\boldsymbol{R}$ and satisfies

$$|g(u)| \le k_1|u|^{\alpha+1} \tag{6.5}$$

and

$$|g'(u)| \le k_1|u|^{\alpha} \tag{6.6}$$

with $k_1 > 0$ and α such that

$$p < \alpha < \frac{8 + (N-2)p^2}{(p+2)(N-4)^+}\left(\le \frac{4}{(N-4)^+}\right). \tag{6.7}$$

Our result reads as follows.

Theorem 3. *Under the hypotheses* Hyp.A′, Hyp.B′ *and* Hyp.C, *there exists a certain open (unbounded) set* $\mathscr{S}$ *in* $H_2 \cap H_1^0 \times H_1^0$ *including* $(0,0)$ *such that if* $(u_0, u_1) \in \mathscr{S}$, *the problem* (P) *admits a solution* $u(t)$ *in the class*

$$L^\infty([0,\infty); H_2 \cap H_1^0) \cap W^{1,2}([0,\infty); H_1^0) \cap W^{1,\infty}([0,\infty); L^2),$$

satisfying a decay estimate

$$E_0(t) = \begin{cases} C_1 e^{-\lambda t} & \text{for some } \lambda > 0 \text{ if } p = 0, \\ C_1(1+t)^{-(p+2)/p} & \text{if } 0 < p \le 4/(N-2)^+. \end{cases}$$

Remark. (1) When $g(u) \equiv 0$ we can take $\mathscr{S} = H_2 \cap H_1^0 \times H_1^0$. (2) Our result is new even for the case $\alpha \le ((N+2)p+4)/(N-p-2)^+$, because our solutions are more regular with respect to x than usual ones. (Cf. [21, 28])

Outline of the proof of Theorem 3.

The proof is given in a parallel way to the one of Theorem 1, and we sketch outline.

First, we note that

$$\|u\|_{\alpha+2}^{\alpha+2} \le C\|\nabla u\|_{p+2}^{(\alpha+2)(1-\theta)}\|\Delta u\|^{(\alpha+2)\theta} \tag{6.8}$$

with

$$\theta = \frac{2(N(\alpha - p) - (\alpha + 2)(p + 2))^+}{(\alpha + 2)(4 - (N - 2)p)} (\leq 1).$$

Therefore, if $\|\Delta u\| \leq K$, we have

$$J(u) \geq k_0 \|\nabla u\|_{p+2}^{p+2} - CK^{(\alpha+2)\theta} \|\nabla u\|_{p+2}^{(\alpha+2)(1-\theta)} \tag{6.9}$$
$$\geq \frac{k_0}{2} \|\nabla u\|_{p+2}^{p+2}$$

provided that

$$CK^{(\alpha+2)\theta} \|\nabla u\|_{p+2}^{(\alpha+2)(1-\theta)-p-2} \leq \frac{k_0}{2}.$$

Then, let us define a stable set $\mathscr{W}_K$ by

$$\mathscr{W}_K \equiv \{u \in H_2 \cap H_1^0 | \|\Delta u\| < K \text{ and } \|\nabla u\|_{p+2} < \varepsilon_0\}, \tag{6.10}$$

where $\varepsilon_0 \equiv \varepsilon_0(K)$ is taken as

$$CK^{(\alpha+2)\theta} \varepsilon_0^{(\alpha+2)(1-\theta)-p-2} = \frac{k_0}{2}. \tag{6.11}$$

We may assume $\tilde{J}(u)$ also satisfies (6.9). Thus, if $u(t)$ is an assumed local strong solution and $u(t) \in \mathscr{W}_K$ for $0 < t \leq T$ with $T > 1$, we can derive the difference inequality (see [18])

$$E(t) \leq C(D(t)E(t)^{1/(p+2)} + D(t)^2)$$

or

$$E(t)^{2(p+1)/(p+2)} \leq C(1 + E(0)^{2p/(p+2)})(E(t) - E(t+1)), \tag{6.12}$$

where we recall

$$\int_t^{t+1} \|\nabla u_t(s)\|^2 ds = E(t) - E(t+1) \equiv D(t)^2.$$

(6.12) implies

$$E(t) \leq g(t) \equiv \{E(0)^{-p/(p+2)} + C^{-1}(1 + E(0))^{-2p/(p+2)}(t-1)^+\}^{-(p+2)/p} \tag{6.13}$$

for $0 < t \leq T$.

To estimate $\|\Delta u(t)\|$ we can utilize again the inequality (4.13). Further, in the case $p < \alpha \le (2p+4)/(N-P-2)^+$, we see

$$(6.14)\quad \int_\Omega |u|^\alpha |\nabla u|^2 dx \le \left(\int_\Omega |u|^{(p+2)N/(N-p-2)} dx\right)^{(N-p-2)\alpha/(p+2)N}$$
$$\times \left(\int_\Omega |\nabla u|^{2(p+2)N/((p+2)N-(N-p-2)\alpha)} dx\right)^{1-(N-p-2)\alpha/(p+2)N}$$
$$\le C\|\nabla u\|_{p+2}^\alpha \|\nabla u\|_{p+2}^{2(1-\theta_1)} \|\Delta u\|^{2\theta_1}$$
$$\le CK^{2\theta_1} E(t)^{(\alpha+2(1-\theta_1))/(p+2)}$$

with

$$\theta_1 = \left(\frac{1}{p+2} - \frac{(p+2)N-(N-p-2)\alpha}{2(p+2)N}\right)^+ \left(\frac{1}{N} + \frac{1}{p+2} - \frac{1}{2}\right)^{-1}$$
$$= \frac{((N-p-2)\alpha - pN)^+}{2p+4-pN} < 1$$

(Some modification is needed if $0 < N \le p+2$). Since

$$\gamma \equiv \frac{\alpha+2(1-\theta_1)}{p+2} \cdot \frac{p+2}{p} = \frac{p+2}{p} \cdot \frac{\alpha(4-N)+4}{4-(N-2)p} > 1,$$

we have

(6.15)
$$\int_0^t \int_\Omega |u|^\alpha |\nabla u|^2 dx ds \le CK^{2\theta_1} \int_0^\infty g(s)^{(\alpha+2(1-\theta_1))/(p+2)} ds$$
$$\le CK^{2\theta_1} (E(0)^{p\gamma/(p+2)} + (1+E(0))^{2p/(p+2)} E(0)^{p(1-\gamma)/(p+2)})$$
$$\equiv q(K, E(0)).$$

When $\alpha > (2p+4)/(N-p-2)^+$, we see

$$(6.16)\quad \int_\Omega |u|^\alpha |\nabla u|^2 dx \le \|u\|_{N\alpha/2}^\alpha \|\nabla u\|_{2N/(N-2)}^2$$
$$\le C\|\nabla u\|_{p+2}^{(1-\theta_2)\alpha} \|\Delta u\|^{\theta_2 \alpha} \|\Delta u\|^2$$
$$\le CK^{2+\theta_2\alpha} g(t)^{(1-\theta_2)\alpha/(p+2)},$$

where

$$\theta_2 = \left(\frac{1}{p+2} - \frac{1}{N} - \frac{2}{N\alpha}\right)\left(\frac{2}{N} + \frac{1}{p+2} - \frac{1}{N} - \frac{1}{2}\right)^{-1}$$
$$= \frac{2(N\alpha - (\alpha+2)(p+2))}{\alpha(2(p+2) - Np)}.$$

Since

$$\gamma \equiv \frac{(1-\theta_2)\alpha}{p+2} \cdot \frac{p+2}{p} = \frac{p+2}{p} \cdot \frac{\alpha(4-N)+4}{4-(N-2)p} > 1,$$

we conclude again (6.15). Thus, we obtain

$$\|\Delta u(t)\|^2 \leq I_0^2 + q(K, E(0)) \equiv Q^2(K, I_0, E(0)). \tag{6.17}$$

Now, defining

$$S_K \equiv \{(u_0, u_1) \in H_1^0 \cap H_2 \times H_1^0 | Q(K, I_0, E(0)) < K \text{ and } (2k_0^{-1}E(0))^{1/(p+2)} < \varepsilon_0(K)\}$$

and taking $(u_0, u_1) \in S_K$, an assumed local strong solution $u(t)$ with $u(0) = u_0$, $u_t(0) = u_1$ exists in fact globally and $u(t) \in \mathscr{W}_K$ for all $t \geq 0$.

Let $u_m(t)$ be the approximate solutions as in Theorem 1. Then, all the estimates in the above are still valid for $u_m(t)$, m large. Noting that if $q < 4/(N-2)^+$,

$$L^\infty_{\text{loc}}([0,\infty); H_1^0 \cap H_2) \cap W^{1,2}_{\text{loc}}([0,\infty); H_1^0)$$

is compactly imbedded in $L^{q+2}_{\text{loc}}([0,\infty); W_0^{1,q+2})$, we conclude that a subsequence of the approximate solutions $u_m(t)$ converges in an appropriate sense to the desired solution $u(t)$ if $(u_0, u_1) \in S_K$. $\mathscr{S}$ is defined of course by $\mathscr{S} \equiv \bigcup_{K>0} S_K$.

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(Ricevita la 13-an de majo, 1996)