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Cauchy Problem for Some Degenerate Abstract Differential Equations of Sobolev Type

By

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1. Introduction

Let H be a Hilbert space over the reals. We shall consider an ordinary differential equation in H:

$$B\frac{du}{dt}(t) + Au(t) \ni f(t), \qquad t \geq 0, \tag{E}$$

together with the initial condition

$$B^{1/2}u(0) = B^{1/2}u_0. \tag{IC}$$

Here A is a nonlinear m-accretive operator in H and B is a nonnegative selfadjoint operator in H.

Equations of the form (E) appear in various physical problems including the propagation of long waves of small amplitude [3], the heat conduction involving two temperatures [7] and soil mechanics [2] and are called pseudo-parabolic equations (since they are parabolic equations when $B = I$) or equations of Sobolev type [12].

In the previous works, the authors have established the existence-uniqueness theorem of strong solutions of the initial value problem for (E) under various conditions on A and B:

i) $D(B) \subset D(A)$ and B has the bounded inverse [13],

ii) $D(B) \subset D(A)$ and B is not necessarily invertible [14],

iii) $D(A) \subset D(B)$ and B has the bounded inverse [8].

The purpose of this paper is to construct a result for (E), (IC) under the case

iv) B is not necessarily invertible (and no inclusion relation between $D(A)$ and $D(B)$ is assumed),

and apply it to the initial-boundary value problem for some nonlinear partial differential equations.

The pseudo-parabolic equation is investigated, in the abstract frame work, by many authors under various conditions on A and B which are different from ours [5], [6], [9], [11], [15] (and the references therein).

2. Notation and a result

Let H be a real Hilbert space, the inner product and the norm in H denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively.

Let A be a nonlinear multivalued operator from H into itself. An operator $A: H \to H$ with domain $D(A) = \{u; Au \neq \phi\}$ and range $R(A) = \cup\{Au; u \in D(A)\}$ is said to be *accretive* if

$$(v_1 - v_2, u_1 - u_2) \geq 0 \qquad \text{for every } v_j \in Au_j \quad (u_j \in D(A), j = 1, 2).$$

An accretive operator A is said to be *m-accretive* if $R(I + A) = H$. For each integer $n > 0$ the *Yosida approximation* A_n of an m-accretive operator A is defined by

$$A_n = n(I - (I + n^{-1}A)^{-1}).$$

It is well known that A_n is m-accretive on H and Lipschitz continuous with n as Lipschitz constant. For brevity of notation, we shall denote $(I + n^{-1}A)^{-1}$ by J_n^A. From the definition

$$(\dagger) \qquad A_n u \in AJ_n^A u \qquad \text{for every } u \in H.$$

We shall denote by Φ the set of all lower semicontinuous (l.s.c.) convex functions from H into $(-\infty, +\infty]$, not identically $+\infty$. For $\varphi \in \Phi$, let $D(\varphi) = \{u \in H; \varphi(u) < +\infty\}$ and denote by $\partial\varphi$ the *subdifferential* of φ:

$$\partial\varphi(u) = \{\xi \in H; \varphi(u) - \varphi(v) \leq (\xi, u - v) \text{ for every } v \in H\}$$

with $D(\partial\varphi) = \{u \in H; \partial\varphi(u) \neq \phi\}$. It is well known that $\partial\varphi$ is m-accretive in H and $D(\partial\varphi)$ is dense in $D(\varphi)$. We refer to [1] and [4] for the properties of m-accretive operators in a Hilbert space.

By $AC([0, T]; H)$ we denote the space of all H-valued strongly absolutely continuous functions on $[0, T]$. For other function spaces, we shall employ the usual notation [10].

Definition 1. An H-valued function $u(t)$ is called a *strong solution* of (E), (IC) if

1) $u \in AC([\delta, T]; H) \quad (\forall\delta > 0), \qquad \frac{du}{dt}(t) \in D(B)$ a.e. $(0, T)$,
2) $u(t) \in D(A)$ a.e. $(0, T)$ and there exists a $\xi(t) \in Au(t)$ such that

$$B\frac{du}{dt}(t) + \xi(t) = f(t) \qquad \text{a.e. } (0, T),$$

3) $B^{1/2}u \in AC([0, T]; H)$ and $B^{1/2}u(t)$ satisfies (IC).

Remark 1. If in addition $Bu \in AC([0, T]; H)$ then

$$\frac{d}{dt}(Bu)(t) = B\frac{du}{dt}(t) \qquad \text{a.e. } (0, T).$$

To establish the existence-uniqueness theorem for (E), (IC), we shall assume the followings.

(A.1) $A = \partial\varphi$, where $\varphi \in \Phi$ and for every $u \in D(\varphi)$

$$\varphi(u) \geq a\|u\|^2 - a' \qquad (a, a' > 0).$$

(A.2) For every $\xi \in Au$ and $\eta \in Av$ $(u, v \in D(A))$ there exists a constant $d > 0$ such that

$$(\xi - \eta, u - v) \geq d\|u - v\|^2.$$

(A.3) B is a nonnegative selfadjoint operator in H.

(A.4) For every $u, v \in D(B)$ and integer $n > 0$

$$(A_n u - A_n v, B(u - v)) \geq 0.$$

Remark 2. Note that

$$(\xi - \eta, B(u - v)) \geq 0 \qquad \text{for every } \xi \in Au \text{ and } \eta \in Av \ (u, v \in D(A) \cap D(B))$$

implies (A.4) because

$$(A_n u - A_n v, B(u - v)) = \frac{1}{n}\|B^{1/2}(A_n u - A_n v)\|^2 + (A_n u - A_n v, B(J_n^A u - J_n^A v)) \geq 0.$$

Here we have used the selfadjointness of B, the indentity

$$(\sharp) \qquad v = n^{-1} A_n v + J_n^A v \ (v \in H)$$

and (†).

Theorem 1. *Assume that* (A.1) $\sim$ (A.4) *are satisfied. Then for every* $0 < T < +\infty$, $f \in W^{1,2}(0, T; H)$ *and* $u_0 \in D(A) \cap D(B)$ *there exists one and only one strong solution* $u(t)$ *of* (E), (IC) *such that*

$$u, \xi (\in Au) \in L^2(0, T; H),$$

$$\sqrt{t}(du/dt) \in L^2(0, T; H), \quad (t > 0)$$

and

$$B^{1/2}u, Bu \in AC([0, T]; H), \qquad (d/dt)(B^{1/2}u), (d/dt)(Bu) \in L^2(0, T; H).$$

3. Proof of Theorem 1

1°. *Uniqueness.* From (A.2) and (A.3) uniqueness of solutions can be obtained by the standard procedure.

2°. *Existence.* Consider an approximate equation for (E):

$$(\mathrm{E}_n) \qquad \left(\frac{1}{n}+B\right)u_n'(t)+A_nu_n(t)=f(t), \qquad t\geq 0,$$

together with

$$(\mathrm{IC}_n) \qquad u_n(0)=u_0.$$

Here $'=d/dt$ and for each integer $n>0$, A_n is the Yosida approximation of A. Since the mapping $v\mapsto (n^{-1}+B)^{-1}A_nv$ from H into H is Lipschitz continuous, the theory of ordinary differential equations in Hilbert space yields that for every $0<T<+\infty$, $f\in L^2(0,T;H)$ and integer $n>0$ there exists a unique function $u_n(\cdot)\in C^1([0,T];D(B))$ which satisfies (E_n) and (IC_n). We shall prove that u_n converges to a solution u of (E), (IC) as n tends to infinity. To this end some a priori estimates are necessary.

i) Taking the inner product of the both sides of (E_n) by $u_n(t)$ we have

$$(3.1) \qquad \frac{1}{2n}\frac{d}{dt}\|u_n(t)\|^2+\frac{1}{2}\frac{d}{dt}\|B^{1/2}u_n(t)\|^2+(A_nu_n(t),u_n(t))=(f(t),u_n(t)).$$

For simplicity, for a function $v(t)\in H$ we shall suppress a letter t and denote by v in several places below. Using the identity (♯), (A.2) and (†) we can estimate the terms in (3.1) as

$$(A_nu_n,u_n)=\frac{1}{n}\|A_nu_n\|^2+(A_nu_n,J_n^Au_n)\geq\frac{1}{n}\|A_nu_n\|^2+d\|J_n^Au_n\|^2,$$

and

$$\begin{aligned}(f,u_n)&\leq\|f\|\left\|\frac{1}{n}A_nu_n+J_n^Au_n\right\|\\&\leq\left(\frac{1}{2n}+c\right)\|f\|^2+\frac{1}{2n}\|A_nu_n\|^2+\frac{d}{2}\|J_n^Au_n\|^2.\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{2n}\frac{d}{dt}\|u_n(t)\|^2+\frac{1}{2}\frac{d}{dt}\|B^{1/2}u_n(t)\|^2+\frac{1}{2n}\|A_nu_n(t)\|^2+\frac{d}{2}\|J_n^Au_n(t)\|^2\\&\leq\left(\frac{1}{2n}+c\right)\|f(t)\|^2.\end{aligned}$$

Here and in the sequel of this paper, by c we denote various positive constants independent of n in a certain interval $[N,+\infty)$ $(N>0)$. Integrating both sides of this inequality over $(0,t)$ and taking the assumptions on $u_n(0)$ and $f(t)$ into account we see

$$\frac{1}{\sqrt{n}}\|u_n(t)\| \le c \qquad \text{for every } t \in [0, T], \tag{3.2}$$

$$\|B^{1/2}u_n(t)\| \le c \qquad \text{for every } t \in [0, T], \tag{3.3}$$

$$\frac{1}{\sqrt{n}}|A_n u_n|_T \le c \tag{3.4}$$

and

$$|J_n^A u_n|_T \le c. \tag{3.5}$$

Here $|\cdot|_T$ is the norm in $L^2(0, T; H)$.

ii) Since u_n belongs to $C^1([0, T]; D(B))$ for each integer $n > 0$

$$\left(\frac{1}{n} + B\right)u_n'(0) + A_n u_n(0) = f(0) \tag{3.6}$$

holds. Taking the inner product of the both sides of (3.6) by $n^{-1}u_n'(0)$ we have

$$\left\|\frac{1}{n}u_n'(0)\right\|^2 + \left\|\frac{1}{\sqrt{n}}B^{1/2}u_n'(0)\right\|^2 \le \frac{1}{2}\|f(0) - A_n u_n(0)\|^2 + \frac{1}{2}\left\|\frac{1}{n}u_n'(0)\right\|^2$$

which implies

$$\left\|\frac{1}{n}u_n'(0)\right\| \le c \tag{3.7}$$

and

$$\left\|\frac{1}{\sqrt{n}}B^{1/2}u_n'(0)\right\| \le c. \tag{3.8}$$

iii) For every $t \in [0, T]$, $h > 0$ and integer $n > 0$

$$\left(\frac{1}{n} + B\right)(u_n'(t+h) - u_n'(t)) + A_n u_n(t+h) - A_n u_n(t) = f(t+h) - f(t) \tag{3.9}$$

holds. Taking the inner product of the both sides of (3.9) by $B(u_n(t+h) - u_n(t))$ and using (A.3) and (A.4) we have

$$\begin{aligned} &\frac{1}{2n}\frac{d}{dt}\|B^{1/2}(u_n(t+h) - u_n(t))\|^2 + \frac{1}{2}\frac{d}{dt}\|B(u_n(t+h) - u_n(t))\|^2 \\ &\quad \le (f(t+h) - f(t), B(u_n(t+h) - u_n(t))). \end{aligned} \tag{3.10}$$

Dividing the both sides of (3.10) by h^2, integrating it over $(0, t)$ and letting h

tends to zero we have

$$\frac{1}{n}\|B^{1/2}u_n'(t)\|^2+\|Bu_n'(t)\|^2\le c\int_0^t\|f'(t)\|^2dt+c\int_0^t\|Bu_n'(t)\|^2dt$$
$$+\|Bu_n'(0)\|^2+\frac{1}{n}\|B^{1/2}u_n'(0)\|^2$$

which implies

$$\|Bu_n'(t)\|\le c\qquad\text{for every } t\in[0,T]\tag{3.11}$$

and

$$\frac{1}{\sqrt{n}}\|B^{1/2}u_n'(t)\|\le c\qquad\text{for every } t\in[0,T].\tag{3.12}$$

Here we have used (3.7), (3.8) and the identity $Bu_n'(0)=-n^{-1}u_n'(0)-A_nu_n(0)+f(0)$. From (3.11) and the assumption $u_0\in D(B)$ we also have

$$\|Bu_n(t)\|\le c\qquad\text{for every } t\in[0,T].\tag{3.13}$$

iv) Taking the inner product of the both sides of (E_n) by $u_n'(t)$ and integrating it over $(0,t)$ we have

$$\int_0^t\left\|\frac{1}{\sqrt{n}}u_n'(t)\right\|^2dt+\int_0^t\|B^{1/2}u_n'(t)\|^2dt+\varphi_n(u_n(t))=\varphi_n(u_0)+\int_0^t(f(t),u_n'(t))dt.$$

Here φ_n is defined by $\varphi_n(v)=(2n)^{-1}\|A_nv\|^2+\varphi(J_n^Av)$ $(v\in H)$ and we have used the fact $(A_nv(t),v'(t))=(d/dt)\varphi_n(v(t))$ ([4], Lemme 3.3). In order to get the boundedness of $|(1/\sqrt{n})u_n'|_T$ we can estimate the right hand side of this equality as

$$\int_0^t(f(t),u_n'(t))dt=(f(t),u_n(t))-(f(0),u_0)-\int_0^t(f'(t),u_n(t))dt$$
$$\le\|f(t)\|\left\|\frac{1}{n}A_nu_n(t)+J_n^Au_n(t)\right\|+\|f(0)\|\|u_0\|$$
$$+\int_0^t\|f'(t)\|\left\|\frac{1}{n}A_nu_n(t)+J_n^Au_n(t)\right\|dt.$$

According to (A.1) the first term of the right hand side of this inequality can be estimated from above by

$$\|f(t)\|\left\|\frac{1}{n}A_nu_n+J_n^Au_n\right\|\le\frac{1}{4n}\|A_nu_n\|^2+\frac{c}{n}\|f(t)\|^2+\frac{a}{2}\|J_n^Au_n\|^2+c\|f(t)\|^2$$
$$\le\frac{1}{2}\left(\frac{1}{2n}\|A_nu_n\|^2+\varphi(J_n^Au_n)\right)+\left(\frac{c}{n}+c\right)\|f(t)\|^2+\frac{a'}{2}$$
$$=\frac{1}{2}\varphi_n(u_n(t))+c\|f(t)\|^2+c.$$

Combining them we get by (3.4) and (3.5)

$$\int_0^t \left\| \frac{1}{\sqrt{n}} u_n'(t) \right\|^2 dt + \int_0^t \|B^{1/2} u_n'(t)\|^2 dt + \frac{1}{2}\, \varphi_n(u_n(t))$$

$$\leq c + \int_0^T \|f'(t)\| \left\| \frac{1}{n}\, A_n u_n(t) + J_n^A u_n(t) \right\| dt$$

$$\leq c + \left(\frac{c}{n} + c\right) |f'|_T^2 + \frac{c}{n} \int_0^T \|A_n u_n(t)\|^2 dt + c \int_0^T \|J_n^A u_n(t)\|^2 dt \leq c.$$

From this inequality and taking $\varphi(J_n^A v) < \varphi_n(v)$ $(v \in H)$ into account, we see

(3.14) $$\|J_n^A u_n(t)\| \leq c \qquad \text{for every } t \in [0, T],$$

(3.15) $$\left| \frac{1}{\sqrt{n}}\, u_n' \right|_T \leq c$$

and

(3.16) $$|B^{1/2} u_n'|_T \leq c.$$

v) Taking the inner product of the both sides of (3.9) by $t(u_n(t+h) - u_n(t))$, $(t > 0)$ we have

$$\frac{1}{2n}\frac{d}{dt} \|\sqrt{t}(u_n(t+h) - u_n(t))\|^2 - \frac{1}{2n} \|u_n(t+h) - u_n(t)\|^2$$

$$+ \frac{1}{2n}\frac{d}{dt} \|\sqrt{t} B^{1/2}(u_n(t+h) - u_n(t))\|^2 - \frac{1}{2} \|B^{1/2}(u_n(t+h) - u_n(t))\|^2$$

$$+ t(A_n u_n(t+h) - A_n u_n(t), u_n(t+h) - u_n(t))$$

$$= t(f(t+h) - f(t), u_n(t+h) - u_n(t)).$$

Dividing the both sides of this equality by h^2, integrating it over $(0, T)$ and letting h tends to zero, we see using (♯), (A.2), (3.15) and (3.16) that

(3.17) $$|\sqrt{t} u_n'|_T \leq c.$$

vi) From (E_n) and the estimates (3.11) and (3.15) we finally obtain

(3.18) $$|A_n u_n|_T = \left| f - B u_n' - \frac{1}{n}\, u_n' \right|_T \leq c$$

and

(3.19) $$|u_n|_T \leq c.$$

Here in (3.19) we have used (3.5), (3.18) and the identity (♯).

From these estimates obtained in the steps i) ~ vi) it follows that a subsequence (denoted again by u_n) can be extracted from $\{u_n\}$ such that as $n \to +\infty$,

$$\begin{aligned}
&u_n \to u && \text{in } L^2(0,T;H) \text{ weakly,}\\
&\frac{1}{n}u_n' \to 0 && \text{in } L^2(0,T;H),\\
&\sqrt{t}u_n' \to \sqrt{t}u' && \text{in } L^2(0,T;H) \text{ weakly,}\\
&B^{1/2}u_n \to B^{1/2}u && \text{in } L^\infty(0,T;H) \text{ weakly star,}\\
&Bu_n \to Bu && \text{in } L^\infty(0,T;H) \text{ weakly star,}\\
&B^{1/2}u_n' \to (B^{1/2}u)' && \text{in } L^2(0,T;H) \text{ weakly,}\\
&Bu_n' \to (Bu)' && \text{in } L^\infty(0,T;H) \text{ weakly star,}
\end{aligned}$$

and

$$A_n u_n \to \xi \qquad \text{in } L^2(0,T;H) \text{ weakly.}$$

Note that $(Bu)'(t) = Bu'(t)$ holds a.e. $(0,T)$ by means of Remark 1.

Now passing to the limit in (E_n) as $n \to +\infty$ we see

$$B\frac{du}{dt}(t) + \xi(t) = f(t) \qquad \text{a.e. } (0,T).$$

Thus for concluding the proof, we need only to show that $\xi(t) \in \partial\varphi(u(t))$ for a.a. $t \in (0,T)$ and $B^{1/2}u(t)$ satisfies (IC).

For every integer $m, n > 0$ we have

$$\left(\frac{1}{m}u_m' - \frac{1}{n}u_n', u_m - u_n\right) + \frac{1}{2}\frac{d}{dt}\|B^{1/2}(u_m - u_n)\|^2 + (A_m u_m - A_n u_n, u_m - u_n) = 0.$$

We see from (3.15) and (3.19) that

$$\left|\int_0^T \left(\frac{1}{m}u_m' - \frac{1}{n}u_n', u_m - u_n\right)dt\right| \le c\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}\right)$$

and from (A.2) and (3.18) that

$$\begin{aligned}
\int_0^t (A_m u_m - A_n u_n, u_m - u_n)dt &= \int_0^t \left(A_m u_m - A_n u_n, \frac{1}{m}A_m u_m - \frac{1}{n}A_n u_n\right)dt\\
&\quad + \int_0^t (A_m u_m - A_n u_n, J_m^A u_m - J_n^A u_n)dt\\
&\ge -c\left(\frac{1}{m} + \frac{1}{n}\right) + d\int_0^t \|J_m^A u_m - J_n^A u_n\|^2 dt.
\end{aligned}$$

Hence combining them we have

$$\frac{1}{2}\|B^{1/2}(u_m(t)-u_n(t))\|^2+d\int_0^t\|u_m(t)-u_n(t)\|^2dt\le c\left(\frac{1}{m}+\frac{1}{n}\right)$$

which follows

(3.20) $$u_n\to u \quad \text{in } L^2(0,T;H) \text{ strongly}$$

and

(3.21) $$B^{1/2}u_n\to B^{1/2}u \quad \text{in } C([0,T];H) \text{ strongly.}$$

Then in virtue of (3.20) and demiclosedness of A, we see $u(t)\in D(A)$ and $\xi(t)\in Au(t)$ for a.a. $t\in(0,T)$. Finally $B^{1/2}u(t)$ satisfies (IC) by means of (3.21). This completes the proof of Theorem 1.

4. Application

We shall apply Theorem 1 to the initial-boundary value problem for some nonlinear partial differential equations.

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$. We shall denote Δ the Laplacian in $\boldsymbol{R}^N$ and $\partial/\partial n$ the outward normal derivative at $\partial\Omega$.

Example 1. Consider an initial-boundary value problem

$$(*)\qquad\begin{cases} -\Delta\dfrac{\partial u(x,t)}{\partial t}+(I-\Delta)u(x,t)=f(x,t) & \text{in } \Omega\times(0,T),\\ \dfrac{\partial u(x,t)}{\partial n}\in-\beta(u(x,t)),\ \dfrac{\partial}{\partial n}\left(\dfrac{\partial u(x,t)}{\partial t}\right)=0 & \text{on } \partial\Omega\times(0,T),\\ (-\Delta)^{1/2}u(x,t)|_{t=0}=(-\Delta)^{1/2}u_0(x) & \text{in } \Omega. \end{cases}$$

Here $x\in\Omega$, $t\in(0,T)$, $0<T<+\infty$ and β is an m-accretive operator in $\boldsymbol{R}$ such that $D(\beta)$ is dense in $\boldsymbol{R}$. (Then there exists a l.s.c. convex function $j\colon\boldsymbol{R}\to(-\infty,+\infty]$ such that $j\not\equiv+\infty$ and $\partial j=\beta$, [1].)

Let $B=-\Delta_B=-\Delta$ with $D(B)=\{u\in H^2(\Omega);(\partial u/\partial n)|_{\partial\Omega}=0\}$, then B is a nonnegative selfadjoint operator in $H\equiv L^2(\Omega)$. Let $\varphi\colon L^2(\Omega)\to(-\infty,+\infty]$ be a function defined by

$$\varphi(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega(|u(x)|^2+|\nabla u(x)|^2)dx+\int_{\partial\Omega}j(u)d\sigma & \text{if } u\in H^1(\Omega), j(u)\in L^1(\partial\Omega)\\ +\infty & \text{otherwise.}\end{cases}$$

Then it is well known [1] that $\partial\varphi(u)=(I-\Delta)u$, $u\in D(\partial\varphi)$, where $D(\partial\varphi)=\{u\in H^2(\Omega);\ \partial u/\partial n\in-\beta(u)\text{ a.e. }\partial\Omega\}$. Set $A=\partial\varphi$. We denote $\Delta_A=I-\partial\varphi=I-A$.

Then applying Theorem 1 to $(*)$ we have the following.

Theorem 2. *Assume $j(u) \geq 0$ a.e. on $\partial\Omega$. Then for every $f \in W^{1,2}(0, T; L^2(\Omega))$ and $u_0 \in D(A) \cap D(B)$ there exists a unique function $u(x, t)$ which satisfies $(*)$ such that*

$$u, \quad (I - \Delta_A)u \in L^2(0, T; L^2(\Omega)), \quad \sqrt{t}\,\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)),$$

and

$$(-\Delta_B)^{1/2}u, \quad -\Delta_B u \in AC([0, T]; L^2(\Omega)),$$

$$\frac{\partial}{\partial t}((-\Delta_B)^{1/2}u), \quad \frac{\partial}{\partial t}(-\Delta_B u) \in L^2(0, T; L^2(\Omega)).$$

Proof. (A.1), (A.2) and (A.3) are clear. Then in order to apply Theorem 1 we only show that (A.4) holds. For every $u, v \in D(A) \cap D(B)$, we have

$$\begin{aligned}
&(Au - Av, B(u - v))_H \\
&= \int_\Omega \{(I - \Delta_A)u(x) - (I - \Delta_A)v(x)\}\{-\Delta_B(u(x) - v(x))\}dx \\
&= \int_\Omega |\nabla(u(x) - v(x))|^2 dx + \int_\Omega \Delta_A u(x)\Delta_B u(x)dx + \int_\Omega \Delta_A v(x)\Delta_B v(x)dx \\
&\quad - \int_\Omega \Delta_A u(x)\Delta_B v(x)dx - \int_\Omega \Delta_A v(x)\Delta_B u(x)dx \\
&\geq \int_\Omega |\nabla(u(x) - v(x))|^2 dx \geq 0,
\end{aligned}$$

which means (A.4) by taking Remark 2 into account.

Example 2. Let $e(x)$ be an element of $L^\infty(\Omega)$ such that $e(x) \geq 0$ a.e. in Ω. Consider an initial-boundary value problem

$$(**) \qquad \begin{cases} e(x)\dfrac{\partial u(x,t)}{\partial t} + (I - \Delta)u(x,t) = f(x,t) & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u(x,t)}{\partial n} \in -\beta(u(x,t)) & \text{on } \partial\Omega \times (0, T), \\ \sqrt{e(x)}u(x,t)|_{t=0} = \sqrt{e(x)}u_0(x) & \text{in } \Omega. \end{cases}$$

Let $Bu = e(x)u$ with $D(B) = L^2(\Omega) \equiv H$ and let $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ be a function defined as in Example 1. Set $A = \partial\varphi$. Then B is a (bounded) non-negative selfadjoint operator in $L^2(\Omega)$ and $A = I - \Delta$ with $D(A) = \{u \in H^2(\Omega);$

$\partial u/\partial n \in -\beta(u)$ a.e. $\partial\Omega\}$ is an m-accretive operator in $L^2(\Omega)$ which satisfies (A.1) and (A.2). Then Theorem 1 is applicable to $(**)$. Note that, in this case, the assumption (A.4) is not necessary because $|B^{1/2}u_n'|_T \leq c$ implies $|Bu_n'|_T \leq c$.

References

[1] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Noordhoff, Leyden, 1976.

[2] Barenblatt, G. I., Zheltov, I. P. and Kochina, I. N., Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, J. Appl. Math. Mech., **24** (1960), 1286–1303.

[3] Benjamin, T. B., Bona, J. L. and Mahoney, J. J., Model equations for long waves in nonlinear dispersive systems, Philos. Trans. Roy. Soc. London, Ser. A, **272** (1972), 47–78.

[4] Brézis, H., *Opérateurs maximaux monotones et semi-groupes de contraction dans les espace de Hilbert,* Math. Studies 5, North-Holland, Amsterdam, 1975.

[5] Brill, H., A semilinear Sobolev equation in a Banach space, J. Differential Equations, **24** (1977), 412–425.

[6] Carrol, R. W. and Showalter, R. E., *Singular and Degenerate Cauchy Problems,* Academic Press, New York/London, 1976.

[7] Chen, P. J. and Gurtin, M. E., On a theory of heat conduction involving two temperatures, Z. Angew. Math. Phys., **19** (1968), 614–627.

[8] Kojo, T. (formerly Matahashi) and Tsutsumi, M., Initial value problem for an abstract differential equation in a Hilbert space, Bull. Sci. Eng. Res. Lab., Waseda Univ., **87** (1979), 64–69.

[9] Lagnese, J. E., General boundary value problems for differential equations of Sobolev type, SIAM J. Math. Anal., **3** (1972), 105–119.

[10] Lions, J. L., *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Paris, 1969.

[11] Showalter, R. E., Degenerate parabolic initial-boundary value problems, J. Differential Equations, **31** (1979), 296–312.

[12] Sobolev, S. L., Some new problems in mathematical physics, Izv. Akad. Nauk, SSSR, **18** (1954), 3–50 (Russian).

[13] Tsutsumi, M. and Kojo, T. (formerly Matahashi), On some nonlinear pseudo-parabolic equations, J. Differntial Equations, **32** (1979), 65–75.

[14] Tsutsumi, M. and Kojo, T. (formerly Matahashi), A certain class of singular nonlinear pseudo-parabolic equations, Funkcialaj Ekvacioj, **22** (1979), 313–325.

[15] Zaidman, S., *Topics in abstract differential equations,* Pitman Research Notes in Math., 304, Longman, 1994.

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