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Instability for Autonomous and Periodic Functional Differential Equations with Finite Delay

By

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Dedicated to Professor T. A. Burton on the occasion of his sixtieth birthday

1. Introduction

Stability has been investigated now for more than a century by numerous authors. The main tool has been Lyapunov's direct method. Since the early 1950's, mathematicians have employed invariance properties of limit sets of autonomous and certain nonautonomous systems of differential equations. See, for example, Barbashin and Krasovskii [1], Hale [9], LaSalle [11, 12, 13] for the earlier papers and Haddock, Krisztin, and Terjéki [7] for a fairly extensive, yet incomplete, bibliography.

Although research related to stability and invariance principles has been thorough, there are fewer results in the literature regarding instability—especially in terms of examples. Standard instability theory for ordinary differential equations (ODEs) can be found in Yoshizawa [15] and Burton [2, 3]. Also, different types of instability for ODEs are defined and studied in LaSalle [12]. A recent paper by Haddock and Ko [6] provides an instability theorem via Lyapunov functions and invariance properties for functional differential equations (FDEs) which are autonomous with finite delay. On the other hand, a few earlier results on instability for (nonlinear) FDEs use Lyapunov functionals, as can be seen, for instance, in Burton [2, 3], and Hale [10]; whereas, results using characteristic equations to obtain stability/instability properties for linear autonomous FDEs can be found in several sources including El'sgol'ts and Norkin [4].

The purpose of this paper is to provide new instability results for both autonomous and periodic FDEs with finite delay. The main approach will incorporate Lyapunov-Razumikhin techniques and invariance properties of limit sets of the equation in question. Section 2 contains the basic and standard notation, and the concept of complete instability is defined. In Section 3 we study the complete instability for autonomous FDEs with finite delay. Section 4 contains an extension of some of the results in Section 3 to periodic FDEs.

Examples will be used throughout the paper to illustrate the main results and concepts. In these examples we provide instability results for certain equations with a single delay, two delays, and distributed delay. Remark 3.3 and the last paragraph of the paper contain statements related to open problems.

2. Notation and preliminaries

In this section, we introduce the basic notation and definitions which will be used throughout. Additional terminology will be provided as needed.

Let $r \geq 0$ be given, and let $C_r = C([-r, 0], \boldsymbol{R}^n)$ denote the space of continuous functions that map $[-r, 0]$ into $\boldsymbol{R}^n$. We assume that C_r is equipped with norm

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \qquad \phi \in C_r,$$

where $|\cdot|$ represents any convenient norm in $\boldsymbol{R}^n$. Let G be an open subset of C_r with $0 \in G$. For $\delta > 0$, let $S(\delta) = \{\phi \in G \colon \|\phi\| < \delta\}$.

If $x \colon [-r, A) \to \boldsymbol{R}^n$ is continuous, $0 < A \leq \infty$, then, for each $t \in [0, A)$, x_t in C_r is defined by

$$x_t(s) = x(t+s), \qquad -r \leq s \leq 0.$$

In this paper, we consider the following finite delay FDEs:

$$x'(t) = f(x_t) \tag{2.1}$$

and

$$x'(t) = p(t, x_t), \tag{2.2}$$

where $f \in C(G, \boldsymbol{R}^n)$, $p \in C(\boldsymbol{R}^+ \times G, \boldsymbol{R}^n)$, and $\boldsymbol{R}^+ = [0, \infty)$. In addition, we assume throughout this paper that

(H) f and p map closed and bounded sets into bounded sets, and p is T periodic in t, i.e., $p(t+T, \phi) = p(t, \phi)$ for all $(t, \phi) \in \boldsymbol{R}^+ \times G$. Also, $f(0) = 0$ and $p(t, 0) = 0$ for all $t \geq 0$ to ensure that (2.1) and (2.2) possess the zero solution.

For the notation and concepts in the following, we will deal only with equation (2.1) and omit the similar details for equation (2.2). We denote by $x(t_0, \phi, f)$ the solution of (2.1) satisfying the condition $x_{t_0} = \phi$, where $t_0 \geq 0$, $\phi \in G$, and let $x(t, t_0, \phi, f) = x(t_0, \phi, f)(t)$ be the corresponding value in $\boldsymbol{R}^n$ of $x(t_0, \phi, f)$ at point t. If the context is clear about t_0 and f, we will simply denote $x(t_0, \phi, f)$ and $x(t, t_0, \phi, f)$ by $x(\phi)$ and $x(t, \phi)$, respectively.

Let ϕ in G be such that $x(t, \phi)$ is defined for all $t \geq 0$. Then, the omega limit set $\Omega[\phi]$ of ϕ with respect to (2.1) is defined by

$$\Omega[\phi] = \{\psi \in C_r\colon x_{t_n}(\phi) \to \psi \text{ for some sequence } \{t_n\} \uparrow \infty\}.$$

A set D in G is said to be *positively invariant* with respect to (2.1) if, for each ϕ in D, $x_t(\phi) \in D$ for all $t \geq 0$ for which $x_t(\phi)$ is defined. D is *invariant* with respect to (2.1) if $x_t(\phi)$ is defined on $[0, \infty)$ for all ϕ in D and $x_t(D) = D$ for all $t \in \boldsymbol{R}^+$. It is known [14] that if $x(t, \phi)$ is bounded on $\boldsymbol{R}^+$, then $\Omega[\phi]$ is nonempty, compact, and invariant.

For a subset H in $\boldsymbol{R}^n$ with $0 \in H$, a *Lyapunov function* V on H is a locally nonnegative *Lipschizian* function defined on $\bar{H}$ so that

(i) $V(0) = 0$ and $V(x) > 0$ for $x \in H \backslash \{0\}$, and

(ii) for any function x which is differentiable at t_0 and $x(t_0) \neq 0$, $\frac{d}{dt} V(x(t_0))$ exists,

where $\bar{H}$ is the closure of H in $\boldsymbol{R}^n$.

For a Lyapunov function V, we define the functional $V'_{(2.1)}\colon G \to \boldsymbol{R}$ by

$$V'_{(2.1)}(\phi) = \overline{\lim_{h \to 0^+}} \frac{V(\phi(0) + hf(\phi)) - V(\phi(0))}{h}.$$

If the context is clear, we will denote this functional by $V'(\phi)$. It is well-known [15] that if $x(\cdot)$ is a solution of (2.1) with $x_{t_0} = \phi$ and V is C^1, then

$$V'(\phi) = \frac{d}{dt} V(x(t))|_{t=t_0} = \sum_{i=1}^{n} \frac{\partial V(x(t_0))}{\partial x_i} f_i(\phi),$$

where x_i and f_i are the i-th components of x and f, respectively.

In the two definitions below we introduce the concepts of *a coherent pair of sets relative to* equation (2.1) and *complete instability* of the zero solution of (2.1) relative to a subset of G. The instability concepts are related to—but not the same as—uniform persistence notions such as those defined in [5].

Definition 2.1. (D, H) is said to be *a coherent pair of sets* relative to (2.1), if the conditions below hold:

(1) D is a closed subset of G and positively invariant with respect to (2.1) such that $0 \in D$;

(2) H is an open subset of $\boldsymbol{R}^n$ so that $0 \in \bar{H}$; and

(3) $\phi \in D$ if the range of ϕ is contained in $\bar{H}$, i.e., $\phi(s) \in \bar{H}$ for all $s \in [-r, 0]$.

Definition 2.2. Let D be a subset of G with $0 \in \bar{D}$. The zero solution of (2.1) is said to be *completely unstable* relative to D if there exists $\varepsilon > 0$, such that for all $\delta \in (0, \varepsilon)$ and each $\phi \in S(\delta) \cap D$, there exist a $t_0 \geq 0$, solution $x(t_0, \phi, f)$ of (2.1), and some $t^* > t_0$ such that $|x(t^*, t_0, \phi, f)| \geq \varepsilon$.

Note that this definition can be adapted easily to apply to any FDE which contains the zero solution.

To conclude this section, we mention that the same concept can be defined for ODEs. We also notice that our definition (for ODEs) differs from that in Burton [2], [3], and those in LaSalle [12]. As simple examples from ODEs, we point out that the zero solution of

$$x' = ax, \qquad a > 0$$

is completely unstable relative to $\{x \in \boldsymbol{R}: x \neq 0\}$; while the zero solution of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is completely unstable relative to $D = \{(x, y) \in \boldsymbol{R}^2: 2x + y \neq 0\}$.

Examples from FDEs appear in subsequent sections.

3. Instability for autonomous FDEs

In this section we consider (2.1), the autonomous system of FDEs with finite delay:

$$x'(t) = f(x_t),$$

with the assumptions in the previous section. An instability theorem for the zero solution of (2.1) was developed in [6]. By investigating the proof in [6] carefully, one finds that the result actually yields complete instability. In this section, we will first state the result from [6] as Theorem 3.1. Then, under weaker assumptions, we shall provide several instability theorems for (2.1), with examples illustrating each result. We emphasize that our results have a wider range of applications and that the result in [6] is a corollary of the theorems presented here.

To begin with, let us quote Theorem 2.1 of [6]:

Theorem 3.1. *Suppose there exists a Lyapunov function $V: \boldsymbol{R}^n \to \boldsymbol{R}^+$ such that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. If either*

(i) $V'_{(2.1)}(\phi) > 0$ *for all* ϕ *in* G *with*

$$V(\phi(0)) = \max_{-r \leq s \leq 0} V(\phi(s)) > 0$$

or

(ii) $V'_{(2.1)}(\phi) > 0$ *for all* ϕ *in* G *with*

$$V(\phi(0)) = \min_{-r \le s \le 0} V(\phi(s)) > 0,$$

then the solution $x = 0$ *is unstable.*

As mentioned before, the conditions of Theorem 3.1 actually yield that the zero solution is completely unstable relative to a certain set. This generalization may be observed from Theorem 3.2 and Theorem 3.3 below.

For a given Lyapunov function V, set

$$P_M(V) = \left\{ \phi \in G \colon V(\phi(0)) = \max_{-r \le s \le 0} V(\phi(s)) > 0 \right\}$$

$$P_m(V) = \left\{ \phi \in G \colon V(\phi(0)) = \min_{-r \le s \le 0} V(\phi(s)) > 0 \right\}.$$

Now we are ready for our first extension.

Theorem 3.2. *Let* (D, H) *be a coherent pair of sets for* (2.1). *Suppose there exists a Lyapunov function* V *on* H *such that either*

(i) $V'_{(2.1)}(\phi) > 0$ *for all* ϕ *in* $D \cap P_M(V)$ *or*

(ii) $V'_{(2.1)}(\phi) > 0$ *for all* ϕ *in* $D \cap P_m(V)$.

Then, the zero solution of (2.1) *is completely unstable relative to* $D \cap P_M(V)$, *or respectively,* $D \cap P_m(V)$.

The proof of this theorem requires only modifications of the proofs of [6, Theorem 2.1] and Theorem 3.3 below, and therefore is omitted. Also, instead of D, it suffices to ask only that $D \cap P_M(V)$ or $D \cap P_m(V)$ positively invariant.

Note that if we take $D = G$ and $H = \mathbf{R}^n$, then the conditions of Theorem 3.2 coincide with those in Theorem 3.1, and we obtain a complete instability result instead of just instability. Also we note that it is sometimes more difficult to find a Lyapunov function defined on $\mathbf{R}^n$ than on a subset of $\mathbf{R}^n$.

The next three examples provide direct applications of Theorem 3.2. Example 3.1, which briefly examines the single delay equation considered in [6], merely illustrates that more information can be provided regarding instability than what is stated in [6]. Examples 3.2 and 3.3 indicate a way to apply Theorem 3.2 to equations with distributed delay. Also, the first two examples follow the idea in [6] by considering the Lyapunov function defined on the whole space.

Example 3.1. Consider

$$x'(t) = ah(x(t)) + bh(x(t-r)), \tag{3.1}$$

where $r \geq 0$, $a+b>0$, $h\colon \boldsymbol{R} \to \boldsymbol{R}$ is continuous and strictly increasing with $h(0)=0$.

Let $V(x)=|x|$. In [6] it is shown that the conditions of Theorem 3.1 hold, and hence the zero solution in unstable. However, if we choose $D=G=C_r$, $H=\boldsymbol{R}$, then the conditions of Theorem 3.2 are fulfilled, and thus the zero solution of (3.1) is completely unstable relative to

$$P_M(V)=D\cap P_M(V)=\left\{\phi\in G\colon |\phi(0)|=\max_{-r\leq s\leq 0}|\phi(s)|>0\right\}$$

or

$$P_m(V)=D\cap P_m(V)=\left\{\phi\in G\colon |\phi(0)|=\min_{-r\leq s\leq 0}|\phi(s)|>0\right\}.$$

See [6] for details.

Example 3.2. Let $r>0$, $h\colon \boldsymbol{R}\to\boldsymbol{R}$ be odd, continuous and increasing with $h(0)=0$, $h(x)\neq 0$ for $x\neq 0$, and $p\colon [-r,0]\to\boldsymbol{R}$ be continuous. Define

$$p_+(s)=\begin{cases} p(s), & \text{if } p(s)\geq 0\\ 0, & \text{if } p(s)<0\end{cases} \quad\text{and}\quad p_-(s)=\begin{cases} 0, & \text{if } p(s)>0\\ p(s), & \text{if } p(s)\leq 0.\end{cases}$$

Then $p(s)=p_+(s)+p_-(s)$. Consider the zero solution of

$$x'(t)=ah(x(t))+\int_{-r}^{0}p(s)h(x(t+s))ds. \tag{3.2}$$

Choose $V(x)=x^2/2$ and $D=G=C_r$. Then

$$\begin{aligned} V'(\phi)&=a\phi(0)h(\phi(0))+\phi(0)\int_{-r}^{0}p(s)h(\phi(s))ds\\ &=a\phi(0)h(\phi(0))+\phi(0)\int_{-r}^{0}p_-(s)h(\phi(s))ds+\phi(0)\int_{-r}^{0}p_+(s)h(\phi(s))ds.\end{aligned}$$

Let $\|\phi\|=|\phi(0)|>0$.

Case (i) If $\phi(0)>0$, then $-\phi(0)\leq\phi(s)\leq\phi(0)$. Also,

$$\begin{aligned} V'(\phi)&\geq a\phi(0)h(\phi(0))-\int_{-r}^{0}p_+(s)ds\phi(0)h(\phi(0))+\int_{-r}^{0}p_-(s)ds\phi(0)h(\phi(0))\\ &=\left(a-\int_{-r}^{0}|p(s)|ds\right)\phi(0)h(\phi(0)),\end{aligned}$$

and

$$V'(\phi) \le a\phi(0)h(\phi(0)) + \int_{-r}^{0} p_+(s)ds\phi(0)h(\phi(0)) - \int_{-r}^{0} p_-(s)ds\phi(0)h(\phi(0))$$

$$= \left(a + \int_{-r}^{0} |p(s)|ds\right)\phi(0)h(\phi(0)).$$

Case (ii) If $\phi(0) < 0$, then $\phi(0) \le \phi(s) \le -\phi(0)$.

$$V'(\phi) \ge a\phi(0)h(\phi(0)) - \int_{-r}^{0} p_+(s)ds\phi(0)h(\phi(0)) + \int_{-r}^{0} p_-(s)ds\phi(0)h(\phi(0))$$

$$= \left(a - \int_{-r}^{0} |p(s)|ds\right)\phi(0)h(\phi(0)),$$

and

$$V'(\phi) \le a\phi(0)h(\phi(0)) + \int_{-r}^{0} p_+(s)ds\phi(0)h(\phi(0)) - \int_{-r}^{0} p_-(s)ds\phi(0)h(\phi(0))$$

$$= \left(a + \int_{-r}^{0} |p(s)|ds\right)\phi(0)h(\phi(0)).$$

Therefore, if $a - \int_{-r}^{0} |p(s)|ds > 0$, then $x = 0$ is completely unstable relative to

$$P_M(V) = D \cap P_M(V) = \left\{\phi \in G\colon |\phi(0)| = \max_{-r \le s \le 0} |\phi(s)| > 0\right\}.$$

Note that, if $a + \int_{-r}^{0} |p(s)|ds < 0$, then $x = 0$ is asymptotically stable (cf. [8]).

Now, let us consider the Lyapunov functions defined only on a subset. Notice that in the following example, we do not require that h be odd.

Example 3.3. Consider (3.2) again. We claim that if

$$a + \int_{-r}^{0} p_-(s)ds > 0,$$

then its zero solution is completely unstable relative to

$$P_M(V) \cap D = \{\phi \in C_r\colon \|\phi\| = \phi(0) > 0 \text{ and } \phi(s) \ge 0,\ -r \le s \le 0\}.$$

Proof. Let $H = \mathbf{R}^+$, $V(x) = x$, and

$$D = \{\phi \in C_r\colon \phi(s) \ge 0 \text{ for all } s \in [-r, 0]\}$$

$$P_M(V) = \{\phi \in C_r\colon \|\phi\| = \phi(0) > 0\}.$$

Then, $P_M(V) \cap D$ is positively invariant relative to (3.2).

In fact, if $x(\cdot)$ is a solution with $x_0 = \phi$ for some $\phi \in P_M(V) \cap D$, then,

$$\begin{aligned}
x'(0) &= ah(x(0)) + \int_{-r}^{0} p(s)h(x(s))ds \\
&= ah(x(0)) + \int_{-r}^{0} p_+(s)h(x(s))ds + \int_{-r}^{0} p_-(s)h(x(s))ds \\
&= ah(\phi(0)) + \int_{-r}^{0} p_+(s)h(\phi(s))ds + \int_{-r}^{0} p_-(s)h(\phi(s))ds \\
&\geq ah(\phi(0)) + \int_{-r}^{0} p_-(s)h(\phi(s))ds \\
&\geq \left(a + \int_{-r}^{0} p_-(s)ds\right)h(\phi(0)) > 0.
\end{aligned}$$

Thus, $x(t)$ is increasing at $t = 0$, which implies that $x_t \in P_M(V) \cap D$ for all t in $[0, \varepsilon)$ and some $\varepsilon > 0$. If there is a $t_0 > 0$ such that $x_{t_0} \notin P_M(V) \cap D$, then, we can find the smallest $t^* > 0$ so that

$$x'(t^*) \leq 0, \qquad x(t^*) = \max_{-r \leq s \leq 0} x(t^* + s) > 0.$$

But, by using $x(t^*) = \max_{-r \leq s \leq 0} x(t^* + s) > 0$, and proceeding as above (for $x'(0) > 0$), it is easy to check that $x'(t^*) > 0$. A contradiction. Thus, $x(t) \in P_M(V) \cap D$ for all $t \geq 0$; i.e., $P_M(V) \cap D$ is positively invariant.

Now,

$$\begin{aligned}
V'(\phi) &= ah(\phi(0)) + \int_{-r}^{0} p(s)h(\phi(s))ds \\
&\geq \left(a + \int_{-r}^{0} p_-(s)ds\right)h(\phi(0)) > 0
\end{aligned}$$

for all ϕ in $P_M(V) \cap D$. Therefore, from Theorem 3.2, $x = 0$ is completely unstable relative to $P_M(V) \cap D = \{\phi \in C_r \colon \|\phi\| = \phi(0) > 0$ and $\phi(s) \geq 0$, $-r \leq s \leq 0\}$. □

Remark 3.1. Recall in Example 3.2, the condition for complete instability is $a - \int_{-r}^{0} |p(s)|ds > 0$. Notice that

$$a - \int_{-r}^{0} |p(s)|ds > 0 \qquad \text{implies} \qquad a + \int_{-r}^{0} p_-(s)ds > 0$$

but not conversely.

The method in Example 3.3 is typical. More specifically, the approach to the claim that $P_M(V) \cap D$ is positively invariant will often be used in later examples with the details being omitted.

In Theorem 3.2, we require that $V'(\phi) > 0$ for all ϕ in $D \cap P_M(V)$ or for all ϕ in $D \cap P_m(V)$. Actually, we can replace D with a subset of D. For example, we have

Theorem 3.3. *Let (D, H) be a coherent pair of sets relative to* (2.1), *and V be a Lyapunov function on H. For a constant $B > 0$, set*

$$V_B = \max_{x \in S_B \cap \bar{H}} V(x) > 0, \qquad \textit{where} \qquad S_B = \{x \in \boldsymbol{R}^n \colon |x| \leq B\},$$

$$P_M^*(V) = P_M^*(V, B) = \left\{\phi \in G \colon 0 < V(\phi(0)) = \max_{-r \leq s \leq 0} V(\phi(s)) < V_B\right\},$$

and

$$D^* = D_B^* = \{\phi \in D \colon 0 < \|\phi\| < B\}.$$

If there is a $B > 0$ such that $V'_{(2.1)}(\phi) > 0$ for all $\phi \in D^* \cap P_M^*(V)$, then $x = 0$ is completely unstable relative to $D^* \cap P_M^*(V)$.

Proof. The idea is similar to that in [6]. Let $\varepsilon \in (0, B)$, $\delta \in (0, \varepsilon)$. Choose $\phi \in D^* \cap P_M^*(V)$ such that $V(\phi(0)) = \max_{-r \leq s \leq 0} V(\phi(s)) > 0$ and $0 < \|\phi\| < \delta$. Then $V'(\phi) > 0$. Now, we shall prove that $|x(t^*, \phi)| = \varepsilon$ for some $t^* > 0$. Suppose not. Then, as in [6], we can conclude that

$$V(x(t, \phi)) \to c \leq V_\varepsilon < V_B \qquad \text{as } t \to \infty.$$

Also, notice that $\Omega[\phi] \neq \emptyset$. In particular, there is some $\psi \in \Omega[\phi]$ and $\{t_n\} \subset \boldsymbol{R}^+$, $t_n \to \infty$ as $n \to \infty$, such that $x_{t_n}(\phi) \to \psi$ as $n \to \infty$. Then, it follows that

$$V(x(t + s, \psi)) \equiv c \qquad \text{on } [-r, 0] \text{ for all } t > 0,$$

which implies

$$V'(\psi) = 0. \tag{$*$}$$

On the other hand, $|x(t, \phi)| < \varepsilon$ implies that $\|\psi\| \leq \varepsilon < B$ and, hence, that $\psi \in D^*$. Also, from

$$x_t(\psi) \in \Omega[\phi] \qquad \text{for all } t \geq 0,$$

we obtain

$$|x(t, \psi)| \leq \varepsilon \qquad \text{for all } t \geq 0.$$

Thus, $x_t(\psi) \in D^*$ for all $t \geq 0$, and further,

$$0 < V(x(t,\psi)) = \max_{-r \le s \le 0} V(x(t+s,\psi)) = c < V_B.$$

Hence, by the assumptions, we have

$$(**) \qquad V'(\psi) > 0.$$

The contradiction between $(*)$ and $(**)$ shows that there is some t^* so that $|x(t^*,\phi)| = \varepsilon$. This completes the proof. □

Example 3.4. Consider

$$(3.3) \qquad x'(t) = -cx(t) + bx(t-r)[1 - x(t)].$$

Suppose $c + |b| < 0$ or $c < 0 < b$. Then $x = 0$ is completely unstable relative to D_0, where

$$D_0 = \begin{cases} C_r & \text{if } b = 0,\ c < 0, \\ \left\{\phi \in C_r \colon 0 < |\phi(0)| = \|\phi\| < -\dfrac{c+|b|}{|b|}\right\} & \text{if } b \ne 0,\ c + |b| < 0, \\ \left\{\phi \in C_r \colon 0 < \phi(s) \le \phi(0) < -\dfrac{c}{b} \text{ for all} \in [-r,0]\right\} & \text{if } c < 0 < b. \end{cases}$$

Indeed, let $H = R$, $D = G = C_r$. First, choose $V(x) = x^2/2$, then

$$V'(\phi) = -c\phi^2(0) + b\phi(0)\phi(-r) - b\phi^2(0)\phi(-r).$$

If $b = 0$, $c < 0$, we have $V'(\phi) \ge 0$ for all $\phi \in G$. The equation reduces to an ODE for which the zero solution is completely unstable relative to $\{x \in \boldsymbol{R} \colon x \ne 0\}$.

If $b \ne 0$, and $c + |b| < 0$, we assume $\|\phi\| = |\phi(0)| > 0$.

Case (1) $b > 0$.

(i) If $\phi(0) > 0$, then $-\phi(0) \le \phi(s) \le \phi(0)$, and

$$\begin{aligned} V'(\phi) &\ge -c\phi^2(0) - b\phi^2(0) - b\phi^3(0) \\ &= (-c - b - b\phi(0))\phi^2(0). \end{aligned}$$

(ii) If $\phi(0) < 0$, then $\phi(0) \le \phi(s) \le -\phi(0)$, and

$$\begin{aligned} V'(\phi) &\ge -c\phi^2(0) - b\phi^2(0) + b\phi^3(0) \\ &= (-c - b + b\phi(0))\phi^2(0). \end{aligned}$$

Thus, $V'(\phi) \ge (-c - b - b|\phi(0)|)\phi^2(0)$.

Case (2) $b < 0$.

As above, we have $V'(\phi) \ge (-c + b + b|\phi(0)|)\phi^2(0)$.

Therefore,

$$V'(\phi) \geq (-c - |b| - |b||\phi(0)|)\phi^2(0).$$

Thus, if we take $D^* = \left\{\phi \in G\colon 0 < |\phi(0)| < -\frac{c+|b|}{|b|}\right\}$, then by Theorem 3.3, the zero solution is completely unstable relative to

$$P_M^*(V) = \left\{\phi \in G\colon 0 < |\phi(0)| = \|\phi\| < -\frac{c+|b|}{|b|}\right\}.$$

Second, in the case that $c < 0 < b$, take $V(x) = |x|$, and

$$D^* = \left\{\phi \in C_r\colon 0 < \phi(s) < -\frac{c}{b} \text{ for all } s \in [-r, 0]\right\}.$$

Then, if $\phi \in D^* \cap P_M(V)$, we have $\phi(0) = \|\phi\| > 0$, $c + b\phi(0) < 0$, and thus

$$\begin{aligned} V'(\phi) &= -c\phi(0) + b\phi(-r) - b\phi(0)\phi(-r) \\ &\geq -c\phi(0) - b\phi^2(0) \\ &= -(c + b\phi(0))\phi(0) > 0. \end{aligned}$$

Hence, from Theorem 3.3, the zero solution is completely unstable relative to $D^* \cap P_M(V)$.

Remark 3.2. It is important to note that, with either $V(x) = |x|$ or $V(x) = x^2/2$, we cannot apply Theorem 3.1 to equation (3.3).

As a final example in this section, we consider a nonlinear scalar autonomous FDE with two delays.

Example 3.5. For equation

$$x'(t) = ah(x(t)) + b_1 h(x(t - r_1)) + b_2 h(x(t - r_2)), \tag{3.4}$$

we assume that $r_1 \geq 0$, $r_2 \geq 0$, a, b_1, and b_2 are constants, $h\colon \boldsymbol{R} \to \boldsymbol{R}$ is continuous, nondecreasing, with $h(0) = 0$, and $h(x) \neq 0$ if $x \neq 0$. Let $V(x) = x^2/2$, and $r = \max\{r_1, r_2\}$.

For the sake of simplicity, let $h(x) = x$. Then,

$$V'(\phi) = a\phi^2(0) + b_1\phi(0)\phi(-r_1) + b_2\phi(0)\phi(-r_2).$$

Case (1) $b_1 b_2 \geq 0$.

(i) $b_1 \leq 0$, $b_2 \leq 0$. Let $\|\phi\| = |\phi(0)| > 0$, $D = G = C_r$.
If $\phi(0) > 0$, then $-\phi(0) \leq \phi(s) \leq \phi(0)$ for all $s \in [-r, 0]$.

$$V'(\phi) \geq a\phi^2(0) + b_1\phi^2(0) + b_2\phi^2(0)$$
$$= (a + b_1 + b_2)\phi^2(0).$$

If $\phi(0) < 0$, then $\phi(0) \leq \phi(s) \leq -\phi(0)$ for all $s \in [-r, 0]$.

$$V'(\phi) \geq a\phi^2(0) + b_1\phi^2(0) + b_2\phi^2(0)$$
$$= (a + b_1 + b_2)\phi^2(0).$$

Therefore, from Theorem 3.2, if $a + b_1 + b_2 > 0$, the zero solution is completely unstable relative to $P_M(V) = \{\phi \in G\colon |\phi(0)| = \|\phi\| > 0\}$.

(ii) $b_1 \geq 0$, $b_2 \geq 0$. Let $|\phi(s)| \geq |\phi(0)| > 0$, for all $s \in [-r, 0]$. Similar to (i), we have

$$V'(\phi) \geq (a + b_1 + b_2)\phi^2(0),$$

and if $a + b_1 + b_2 > 0$, the zero solution is completely unstable relative to $P_m(V) = \{\phi \in G\colon |\phi(0)| = \min_{-r \leq s \leq 0} |\phi(s)| > 0\}$.

Case (2) $b_1 b_2 < 0$.

(i) $a - |b_1| - |b_2| > 0$. Let $\|\phi\| = |\phi(0)| > 0$. Similar to (1-i), we have

$$V'(\phi) \geq (a - |b_1| - |b_2|)\phi^2(0),$$

and thus, if $a - |b_1| - |b_2| > 0$, the zero solution is completely unstable relative to $P_M(V) = \{\phi \in G\colon |\phi(0)| = \|\phi\| > 0\}$.

(ii) $b_1 < 0$, $b_2 > 0$, $b_1 + b_2 \leq 0$, $a + b_1 + b_2 > 0$, and $r_1 \geq r_2$. Set

$$D = \{\phi \in C_r\colon \phi \nearrow, \ \phi(-r) \geq 0, \ \phi(0) > 0\}.$$

Then $D \cap P_M(V) = D$ is positively invariant. In fact, if $x(\cdot)$ is a solution with $x_0 = \phi \in D$, then $x(0) \geq x(s) \geq 0$ for all s in $[-r, 0]$. Since x_0 is increasing, we have

$$\begin{aligned} x'(0) &= ax(0) + b_1 x(-r_1) + b_2 x(-r_2) \\ &\geq ax(0) + b_1 x(-r_1) + b_2 x(-r_1) \\ &= ax(0) + (b_1 + b_2)x(-r_1) \\ &\geq (a + b_1 + b_2)x(0) > 0. \end{aligned}$$

Therefore, $x(\cdot)$ is strictly increasing around $t = 0$. Thus, there exists an $\varepsilon > 0$ such that $x_t \in D$ for all $t \in [0, \varepsilon)$. If there is some $t_0 > 0$ so that $x_{t_0} \notin D$, then, we can find the smallest $t^* > 0$ such that

$$x'(t^*) \leq 0, \qquad x(t^*) = \max_{-r \leq s \leq 0} x(t^* + s) > 0.$$

But

$$\begin{aligned} x'(t^*) &= ax(t^*) + b_1x(t^* - r_1) + b_2x(t^* - r_2) \\ &\geq ax(t^*) + b_1x(t^* - r_1) + b_2x(t^* - r_1) \\ &= ax(t^*) + (b_1 + b_2)x(t^* - r_1) \\ &\geq ax(t^*) + (b_1 + b_2)x(t^*) \\ &= (a + b_1 + b_2)x(t^*) > 0. \end{aligned}$$

A contradiction. Hence $x_t \in D$ for all $t > 0$, and D is positively invariant.

Now for all $\phi \in D$,

$$\begin{aligned} V'(\phi) &= a\phi^2(0) + b_1\phi(0)\phi(-r_1) + b_2\phi(0)\phi(-r_2) \\ &\geq a\phi^2(0) + b_1\phi(0)\phi(-r_1) + b_2\phi(0)\phi(-r_1) \\ &= a\phi^2(0) + (b_1 + b_2)\phi(0)\phi(-r_1) \\ &\geq a\phi^2(0) + (b_1 + b_2)\phi^2(0) \\ &= (a + b_1 + b_2)\phi^2(0). \end{aligned}$$

Therefore, by Theorem 3.3, the zero solution is completely unstable relative to

$$D = \{\phi \in C_r\colon \phi \nearrow,\ \phi(-r) \geq 0,\ \phi(0) > 0\}.$$

In summary, each of the following conditions implies complete instability of the zero solution of (3.4):

(1) $b_1b_2 \geq 0$, $a + b_1 + b_2 > 0$, and any $r_1, r_2 \geq 0$ (relative to either $\{\phi \in G\colon |\phi(0)| = \|\phi\| > 0\}$ or $\{\phi \in G\colon |\phi(0)| = \min_{-r \leq s \leq 0} |\phi(s)| > 0\}$).

(2) $b_1b_2 < 0$, $a - |b_1| - |b_2| > 0$, and any $r_1, r_2 \geq 0$ (relative to $\{\phi \in G\colon |\phi(0)| = \|\phi\| > 0\}$).

(3) $b_1b_2 < 0$, $b_1 + b_2 \leq 0$, $a + b_1 + b_2 > 0$, and either $b_1 < 0$, $r_1 \geq r_2$, or $b_1 > 0$, $r_1 \leq r_2$ (relative to $D = \{\phi \in C_r\colon \phi \nearrow,\ \phi(-r) \geq 0,\ \phi(0) > 0\}$).

Remark 3.3. We conjecture that if $a + b_1 + b_2 > 0$, then the zero solution of (3.4) has a complete instability property. However, we have been unable to use our results to prove it at this stage, and leave it as an open problem to the readers.

Remark 3.4. In the linear case, i.e., $h(x) = x$, it is known [4, pp. 139–140] that if $a + b_1 + b_2 > 0$, then for sufficiently small $r = \max\{r_1, r_2\}$, the zero solution of (3.4) is unstable. Note that this is proved by employing the corresponding characteristic equation. The obvious differences between this result and ours are the following: linear vs. nonlinear; dependent on vs. independent of the size of delays; and instability vs. complete instability.

4. Instability for periodic FDEs

In the previous section various theorems and examples were furnished for autonomous FDEs. It is natural to ask, then, "Can these notions be extended to include nonautonomous systems?" The next simple example tells us that, in general, the answer is "Not without modifications."

Example 4.1. Consider a scalar ODE

$$x'(t) = \frac{1}{(t+1)^2}x(t). \tag{4.1}$$

Let $V(x) = x^2/2$. Then, $V'(t,x) = x^2/(t+1)^2 > 0$ if $x \neq 0$. But $x = 0$ is stable for equation (4.1).

Although our autonomous instablility results cannot be extended directly to include nonautonomous cases, there are certain natural expansions to periodic systems. In this section, we consider the T-periodic system (2.2)

$$x'(t) = p(t, x_t)$$

with the appropriate assumptions for (2.2) given in Section 2.

For $\phi \in G$, the omega limit set $\Omega[\phi]$ is defined as before. In addition, we introduce a subset of $\Omega[\phi]$ as follows:

$$\Gamma[\phi] = \{\psi \in C_r : \exists \{m_j\} \subset \boldsymbol{N}, m_j \to \infty \text{ as } j \to \infty \text{ so that } x_{m_jT}(\phi) \to \psi \text{ as } j \to \infty\},$$

where $\boldsymbol{N}$ is the set of all nonnegative integers. Then, if the solution $x(\phi)$ is bounded in the future, $\Gamma[\phi]$ is nonempty, compact, and positively invariant in the discrete sense, that is,

$$x_{jT}(\psi) \in \Gamma[\phi] \qquad \text{for all } j \in \boldsymbol{N} \text{ whenever } \psi \in \Gamma[\phi].$$

Actually, $\Omega[\phi]$ is nonempty, compact, and positively invariant in the discrete sense. Further, it can be proved that the following equality holds:

$$\Omega[\phi] = \bigcup_{t \geq 0} \{x_t(c) : \psi \in \Gamma[\phi]\}. \tag{4.2}$$

For all of these, we refer to LaSalle [13].

The derivative of a Lyapunov function $V(x)$ related to (2.2) is defined by:

$$V'_{(2.2)}(t,\phi) = \overline{\lim_{h \to 0^+}} \frac{V(\phi(0) + hp(t,\phi)) - V(\phi(0))}{h}.$$

For all other notation see Section 2.

Theorem 4.1. *Suppose that there exists a Lyapunov function $V: \boldsymbol{R}^n \to \boldsymbol{R}^+$ such that $V(0) = 0$, $V(x) > 0$ if $x \neq 0$. If*

$$V'_{(2.2)}(t, \phi) > 0 \qquad \text{for all } t \geq 0 \text{ and } \phi \in P_M(V),$$

then the zero solution $x = 0$ of (2.2) is completely unstable relative to $P_M(V)$.

Proof. We use an approach similar to that in the proof of Theorem 3.3. Let $\varepsilon > 0$, $\delta \in (0, \varepsilon)$, $\phi \in P_M(V)$ with $0 < \|\phi\| < \delta$. Then, $V'(t, \phi) > 0$. If for all $t \geq 0$, $|x(t, \phi)| < \varepsilon$, then $x(t, \phi)$ and $x'(t, \phi)$ are bounded on $[0, \infty)$, and it follows that

$$\lim_{t \to \infty} V(x(t, \phi)) = c \qquad \text{for some } c \in \boldsymbol{R}^+,$$

and that $\Omega[\phi] \neq \emptyset$, $\Gamma[\phi] \neq \emptyset$.

Clearly, ψ in $\Omega[\phi]$ implies that

$$V(\psi(s)) \equiv c \qquad \text{on } [-r, 0].$$

Now by equality (4.2),

$$x_{t_0}(\psi) \in \Omega[\psi] \qquad \text{for all } \psi \in \Gamma[\phi] \text{ and all } t_0 \geq 0.$$

Hence, if $\psi \in \Gamma[\phi]$, then

$$V(x(t_0 + s, \psi)) \equiv c \qquad \text{on } [-r, 0] \text{ for all } t_0 \geq 0.$$

Thus, $V'(t, \psi) = 0$ for all $t \geq 0$, a contradiction, since by assumption, we should have $V'(t, \psi) > 0$. Therefore, there exists some $t_0 > 0$ such that $|x(t_0, \phi)| \geq \varepsilon$, and thus, $x = 0$ is completely unstable relative to $P_M(V)$. This completes the proof. $\square$

The following example shows that we cannot change the condition on the derivative in Theorem 4.1 to

$$V'_{(2.2)}(t, \phi) > 0 \qquad \text{for some } t \geq 0 \text{ and all } \phi \in P_M(V).$$

Example 4.2. The general solution of

$$x' = \left(\tfrac{1}{2} - \sin^2 t\right)x$$

is $x(t) = x_0 e^{\sin 2t/2}$, and hence $x = 0$ is stable. But for $V(x) = x^2/2$, we have

$$V'(t, x) = \left(\tfrac{1}{2} - \sin^2 t\right)x^2 > 0 \qquad \text{for infinitely many } t > 0.$$

However, as in the autonomous case, we can weaken the conditions in Theorem 4.1 as:

Theorem 4.2. *Let (D, H) be a coherent pair of sets for* (2.2). *Suppose there exists a Lyapunov function V on H such that*

$$V'_{(2.2)}(t, \phi) > 0 \qquad \text{for all } \phi \in D \cap P_M(V) \text{ and all } t \geq 0.$$

Then, the zero solution of (2.2) *is completely unstable relative to $D \cap P_M(V)$.*

To prove this theorem, we only need to modify slightly the proof of Theorem 4.1. Also, we need only require that $D \cap P_M(V)$ rather than D be positively invariant. The following examples illustrate applications of the theory developed in this section, and provide certain extension of Examples 3.1, 3.2, and 3.3.

Example 4.3. Consider

$$x'(t) = q(t)h(x(t)) + p(t)h(x(t - r)), \tag{4.3}$$

where $h: \boldsymbol{R} \to \boldsymbol{R}$ is continuous and strictly increasing with $h(0) = 0$, $p(t)$, and $q(t)$ are continuous and of period T. If

$$q(t) + p_-(t) > 0 \qquad \text{for all } t \in \boldsymbol{R}^+,$$

then the zero solution $x = 0$ is completely unstable relative to

$$D' = \{\phi \in C_r: \phi(s) \geq 0 \text{ for } s \in [-r, 0], \phi(0) = \|\phi\| > 0\},$$

where p_-, and later, p_+, are defined in the same way as in Example 3.2.

In fact, if we take $H = \boldsymbol{R}^+$, $D = \{\phi \in C_r: \phi(s) \geq 0 \text{ for } s \in [-r, 0]\}$, and $V(x) = x$, then $D' = D \cap P_M(V)$. As in Example 3.3, we can prove that D' is positively invariant.

Further, for $\phi \in D'$,

$$\begin{aligned} V'(t, \phi) &= q(t)h(\phi(0)) + p(t)h(\phi(-r)) \\ &= q(t)h(\phi(0)) + p_+(t)h(\phi(-r)) + p_-(t)h(\phi(-r)) \\ &\geq q(t)h(\phi(0)) + p_-(t)h(\phi(-r)) \\ &\geq q(t)h(\phi(0)) + p_-(t)h(\phi(0)) \\ &= (q(t) + p_-(t))h(\phi(0)) > 0. \end{aligned}$$

Therefore, from Theorem 4.2, we conclude that $x = 0$ is completely unstable relative to D'.

Example 4.4. Let $a(\cdot)$ and $p(\cdot, s)$ be continuous functions of period T for any s in $[-r, 0]$, where $r > 0$. Consider

$$x'(t) = a(t)x(t) + \int_{-r}^{0} p(t, s)x(t + s)ds. \tag{4.4}$$

If $a(t) + \int_{-r}^{0} p_-(t,s)ds > 0$ for all t in $[0,T]$, then the zero solution of (4.4) is completely unstable relative to $P_M(V) \cap D = \{\phi \in C_r: \|\phi\| = \phi(0) > 0$ and $\phi(s) \geq 0, -r \leq s \leq 0\}$.

Proof. Use the same method as in Example 3.3 and Theorem 4.2. □

As a final remark, we mention that, at this time, we have been unable to prove any complete instablity result for the periodic FDEs with finite delay by considering the set $P_m(V)$ instead of $P_M(V)$, as we did in Section 3 for the autonomous FDEs. Such a result, of course, would provide more flexibility in dealing with examples.

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