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A Note on the Nirenberg Example

By

Haruki NINOMIYA

(Osaka Institute of Technology, Japan)

Let X be a nowhere-zero C^{∞} complex vector field in the plane. We know that, by a smooth regular change of coordinates, the homogeneous equation $Xu=0$ in a neighborhood of a point P in $\boldsymbol{R}^2$ is transformed into the form of $Lu \equiv \partial u/\partial t + ia(t,x)\partial u/\partial x = 0$ in a neighborhood of the origin, where $a(t,x)$ is a real-valued C^{∞} function. Does the $Lu=0$ have a non-trivial solution in a neighborhood of the origin? This is positive if $a(t,x)$ is real-analytic with respect to the variable x (due to Nagumo's famous theorem [1]) or $a(0,0) \neq 0$ holds; moreover we know that it is positive when, taken a neighborhood ω of the origin so small if necessary, the function $t \to a(t,x)$ does not change sign in $\{t; (t,x) \in \omega\}$ for every x in $\boldsymbol{R}$ (see Treves [6]). The last condition is a necessary and sufficient condition for the L to be a solvable operator at the origin, and hence it is the problem when L is an unsolvable operator. We are thus concerned with the operator L satisfying $ta(t,x) > 0$ for $t \neq 0$.

Now Nirenberg [4] showed the complex vector field L_0 that any C^1 solution u to the equation $L_0 u = 0$ in a neighborhood of the origin is identically constant. The vector field L_0 has the form of $L_0 = \partial/\partial t + it\{1 + t\phi(t,x)\}\partial/\partial x$, where $\phi(t,x)$ is a non-negative C^{∞} function having the following properties (1) and (2):

(1) $\phi(-t,x) = \phi(t,x)$.

(2) $\phi(t,x)$ is positive inside a sequence of the discs $D_j^{m,n}$ and vanishes outside their union.

where, for positive integers m, n, and j, $D_j^{m,n}$ denote non-overlapping closed discs in the (t,x) plane that satisfy the following for each fixed (m,n):

(i) The ordinates of the centers of $D_j^{m,n}$ equal $1/n$.

(ii) $1/m < t < 1/(m-1)$ for every (t,x) in $D_j^{m,n}$ $(j = 1,2,\dots)$.

(iii) The abscissae of the centers of $D_j^{m,n}$ decrease to $1/m$ as $j \to \infty$.

In this note we shall show that, assuming $ta(t,x) > 0$ for $\neq 0$, the method of the proof of Theorem A in my paper [2] can be applied to give a necessary condition for the equation $Lu=0$ to have a non-trivial solution or a non-constant one in a neighborhood of a point P on the x axis.

To state our result we need a notion of flag domain ([2]):

A domain D in the (t, x) plane is called a flag domain if $D \subset \{(t, x); t>0\}$ and the boundary ∂D of D is a simple closed curve such that $\partial D \cap \{(t, x); t=0\}$ is a line segment with positive length. Let us denote by a_{even} (t, x) the even part of $a(t, x)$ with respect to t.

We obtain the following

Theorem. *Let us assume that $ta(t, x)>0$ for $t \neq 0$. In order that the equation $Lu=0$ has a non-trivial C^1 solution u in a neighborhood of a point P on the axis, it is necessary that every neighborhood U of P contains a flag domain D_0 such that for every flag domain D in D_0 either the boundary $\partial[D \cap \{a_{\text{even}}(t, x)>0\}]$ of $D \cap \{a_{\text{even}}(t, x)>0\}$ or the boundary $\partial[D \cap \{a_{\text{even}}(t, x)<0\}]$ of $D \cap \{a_{\text{even}}(t, x)<0\}$ is not contained in D or not a finite number of rectifiable Jordan curves.*

Proof. We shall prove that the contraposition holds. We may assume that there is a neighborhood U of P such that for every flag domain D_0 in U there exists a flag domain D in D_0 having the property that $\partial[D \cap \{a_{\text{even}}(t, x)>0\}]$ is a finite number of rectifiable Jordan curves in D. Assuming that $Lu=0$ has a non-trivial C^1 solution u in a neighborhood V of P, we show a contradiction. Without loss of generality, we may assume that the neighborhood V is equal to U. First, assume that $u_x(0, x) \equiv 0$. Set $v \equiv u_x$. Note that u and hence v is infinitely many times differentiable in $t \neq 0$, since L is elliptic for $t \neq 0$. Then v is a C^1 solution of $Lv+ia_x v=0$ in $U_{\pm}=U \cap \{t \neq 0\}$ with the initial data 0 on $t=0$. Applying uniqueness theorem (see [3] or [5]), we see that v vanishes identically in U because of the ellipticity of L in $t \neq 0$. But this contradicts the fact that u is not a constant. Thus there exists a real value x_0 such that $u_x(0, x_0) \neq 0$. Then there is a flag domain D_0 in U such that $u_x \neq 0$ in a neighborhood $U_0 \equiv D_0 \cup \{(t, x); (-t, x) \in D_0\} \cup [\bar{D}_0 \cap \{t=0\}]$ of $P_0(0, x_0)$. From our assumption, there is a flag domain D in D_0 such that $\partial[D \cap \{a_{\text{even}}(t, x)>0\}]$ is a finite number of rectifiable Jordan curves in D. Hereafter the same method of the proof of Theorem A in Ninomiya [2] can be applied to get a contradiction. The proof is as follows:

First we can assume that both of Re $\partial u/\partial x$ and Im $\partial u/\partial x$ are positive in U_0. Furthermore we can assume that $D \cap \{a_{\text{even}}(t, x)>0\}$ is an open set ω in U_0 obtained by removing a finite number of simply connected domains or that of multiply connected domains that are disjoint each other from a simply connected domain Ω surrounded by a rectifiable Jordan curve.

Let us denote by a_{odd}, u_{odd}, and u_{even} the odd part of $a(t, x)$ with respect to t, that of $u(t, x)$ with respect to t, and the even part of $u(t, x)$ with respect to t, respectively. From $Lu=0$, we have

$$\partial u_{\text{odd}}/\partial t + i a_{\text{odd}}(t, x) \partial u_{\text{odd}}/\partial x = -i a_{\text{even}}(t, x) \partial u_{\text{even}}/\partial x \tag{1.1}$$

in U_0. Hence, it follows that

$$\partial u_{\text{odd}}/\partial t + ia_{\text{odd}}(t,x)\partial u_{\text{odd}}/\partial x = 0 \qquad \text{in } D \cap \Omega^c \tag{1.2}$$

By our assumption $ta(t,x) > 0$ for $t \neq 0$, we see that $a_{\text{odd}}(t,x) > 0$ for $t > 0$.

Now we note that $u_{\text{odd}}(0,x) \equiv 0$. Therefore, applying uniqueness theorem (see [3] or [5]) to (1.2), we see that $u_{\text{odd}}(t,x)$ vanishes identically in $D \cap \Omega^c$. Now we know that there exists a C^1 solution $v = v(t,x)$ of

$$\partial v/\partial t + ia_{\text{odd}}(t,x)\partial v/\partial x = 0 \tag{1.3}$$

in a neighborhood of P_0 such that $\partial v/\partial x \neq 0$ ([2]). Then, we can assume that v satisfies (1.3) in U_0 and that both of $\operatorname{Re} \partial v/\partial x$ and $\operatorname{Im} \partial v/\partial x$ are positive in U_0. From (1.1), we have

$$(\partial v/\partial x)\{\partial u_{\text{odd}}/\partial t + ia_{\text{odd}}(t,x)\partial u_{\text{odd}}/\partial x\} = (\partial v/\partial x)\{-ia_{\text{even}}(t,x)\partial u_{\text{even}}/\partial x\} \tag{1.4}$$

in U_0. Hence we have

$$\begin{aligned} &\int_\Omega (\partial v/\partial x)\{\partial u_{\text{odd}}/\partial t + ia_{\text{odd}}(t,x)\partial u_{\text{odd}}/\partial x\} dtdx \\ &\quad = \int_\Omega (\partial v/\partial x)\{-ia_{\text{even}}(t,x)\partial u_{\text{even}}/\partial x\} dtdx. \end{aligned} \tag{1.5}$$

From (1.3), we see that the left-hand side of (1.5) $=$

$$\begin{aligned} \int_\Omega \{\partial u_{\text{odd}}/\partial t \cdot \partial v/\partial x - \partial u_{\text{odd}}/\partial x \cdot \partial v/\partial t\} dtdx &= \int_\Omega d\{u_{\text{odd}}(t,x) dv(t,x)\} \\ &= \int_{\partial\Omega} u_{\text{odd}}\{(\partial v/\partial t)dt + (\partial v/\partial x)dx\} \\ &= 0 \end{aligned}$$

because of $u_{\text{odd}} \equiv 0$ on the boundary of Ω.

Therefore, we have

$$\int_\Omega (\partial v/\partial x)\{a_{\text{even}}(t,x)\partial u_{\text{even}}/\partial x\} dtdx = 0.$$

But this contradicts the fact that

$$\begin{aligned} \operatorname{Im}[(\partial v/\partial x)\{a_{\text{even}}\partial u_{\text{even}}/\partial x\}] = a_{\text{even}}(t,x) \cdot \{&\operatorname{Re} \partial u_{\text{even}}/\partial x \cdot \operatorname{Im} \partial v/\partial x \\ &+ \operatorname{Im} \partial u_{\text{even}}/\partial x \cdot \operatorname{Re} \partial v/\partial x\} \end{aligned}$$

is positive in $\omega \subset \Omega$. Q.E.D.

Remark 1. We can easily verify that the Nirenberg example does not satisfy the necessary condition stated in the above Theorem, if we take P as the origin.

Remark 2. When $a(t,x)$ is real analytic with respect to t and x or $\operatorname{supp} a_{\text{even}} = \varnothing$, namely, $a_{\text{even}}(t,x)$ vanishes identically, we know that the necessary condition in Theorem is satisfied and that there exists a non-trivial C^1 solution u of the equation $Lu = 0$ in a neighborhood of P. Hence the interesting case is that $a(t,x)$ belongs to $C^\infty \backslash C^\omega$ and $\operatorname{supp} a_{\text{even}} \neq \varnothing$. I tried in vain to find an interesting example in such a function class that satisfies the necessary condition in Theorem and that there exists a non-trivial C^1 solution u of the equation $\partial u/\partial t + ia(t,x)\partial u/\partial x = 0$ in a neighborhood of P.

Remark 3. Let Y_α denote a C^∞ complex vector field in $\boldsymbol{R}^2$ such that $L_\alpha \equiv \partial/\partial t + i\alpha(t,x)\partial/\partial x$, where $\alpha(t,x)$ is a real-valued C^∞ function. Then there is a question: does there exist an equation $L_\alpha u = 0$ such that the equation $L_\alpha u = 0$ has a non-trivial C^1 solution u in a neighborhood of the origin and any C^1 solution u of the $L_\alpha u = 0$ in a neighborhood of the origin satisfies $du = 0$ at the origin? This is negative if $\alpha(t,x)$ is real analytic or $\alpha(0,0) \neq 0$. It is generally yet open to be solved.

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Department of Mathematics
Osaka Institute of Technology
5–16–1 Omiya Asahi-ku, Osaka 535
Japan

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