Existence of Exactly $N$ Periodic Solutions for Liénard Systems

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§ 1. Introduction

We are interested in the study of periodic solutions of Liénard systems:

$$\begin{cases}
\dot{x} = y - F(x), \\
\dot{y} = -g(x),
\end{cases}$$

where $(\cdot) := (d/dt)$.

For it, there appeared many studies of treating a single periodic solution, however, those of treating several periodic solutions are scanty. In the direction, Rychkov [Ryl] found an intelligible theorem for assuring several periodic solutions. Recently, Alsholm [A1] improved his theorem as follows.

**Theorem A** [A1]. Suppose that System (L) satisfies the following:

(C1) The functions $F, g \in C^0(\mathbb{R})$ are continuous and odd, and $\text{sgn } g(x) = \text{sgn } x$.

(C2) The function $F$ has $N$ many positive zeros

$$(\beta_0 := 0 <) \quad \beta_1 < \beta_2 < \cdots < \beta_N \quad (\beta_{N+1}: \text{a bound})$$

at which it changes its sign alternately, and for each $1 \leq k \leq N$, there is a $C^1$ mapping $\phi_k: [\beta_{k-1}, \beta_k] \to [\beta_k, \beta_{k+1}]$, not necessarily onto, such that

$$g(\phi_k(x))\phi_k'(x) \geq g(x) \quad \text{and} \quad |F(\phi_k(x))| \geq |F(x)|.$$ 

Then it has at least $N$ periodic solutions, all of whose amplitudes $\rho$ are estimated by $\beta_k < \rho < \delta_{k+1} := \phi_k(\beta_k)$ for some $1 \leq k \leq N$.

Here we say that a value $\rho$ is the amplitude of a periodic solution if it is the maximum of the coordinate $x$ of the solution.

On the other hand, by adding some conditions to those of Rychkov's theorem, Ding [Di] made a theorem which assures exactly $N$ periodic solutions. Our purpose is to improve his theorem as follows.

**Theorem B.** Suppose that System (L) satisfies (C1)–(C2) and the following:

(C3) The functions $F, g \in C^1(\mathbb{R})$, and on each interval $[\beta_{k-1}, \delta_k]$, $2 \leq k \leq N+1$, the function $F$ has an extremum at a unique point $\varsigma_k$. 
(C4) The function $g(x) = x$, and on each interval $[\alpha_k, \delta_k]$, $2 \leq k \leq N + 1$, the derivative $f' := F'$ is weakly monotone. Then it has exactly $N$ periodic solutions, all coarse.

Here we say that a periodic solution is coarse, or hyperbolic, if it can not be bifurcated by any $C^1$ perturbations.

When Rychkov and Ding made earlier versions of our theorems, they took the mappings $\phi_k$ as parallel translations. By getting Alsholm's improvement, the theorems have become to contain uncertain mappings, and so we must encounter the choice of mappings. However, when the system satisfies (C3), we can determine the best choice.

**Choice Lemma.** Consider System (L) with (C1)–(C3). Then the mapping

$$
\psi_k(x) = G^{-1}(G(x) + c_k), \quad \text{where } G(x) := \left(\int_0^x gdx\right) \cdot \text{sgn } x.
$$

is the best choice for (C2).

By using Choice Lemma, we can easily check whether System (L) satisfies (C2) or not. In §5, we will learn how to use the mappings. To see the effectiveness of our theorems, we shall show three examples.

**Example 1.** Consider the following system:

(Ex1) \[
\begin{cases}
\dot{x} = y - \mu \sin(|x|^v \text{ sgn } x), \\
\dot{y} = -x,
\end{cases}
\quad \text{where } \mu \neq 0.
\]

If the parameter $0 < v \leq 2$, then in each strip $|x|^v \leq (N + 1)\pi$, it has exactly $N$ periodic solutions.

**Example 2.** Consider Van der Pol system:

(Ex2) \[
\begin{cases}
\dot{x} = y - \mu(x^3/3 - x), \\
\dot{y} = -x,
\end{cases}
\quad \text{where } \mu \neq 0.
\]

Then it has a unique periodic solution, whose amplitude $\rho$ is estimated by $\sqrt{3} \approx 1.7321 < \rho < \sqrt{3 + 2\sqrt{3}} \approx 2.5421$.

**Example 3.** Consider the following system:

(Ex3) \[
\begin{cases}
\dot{x} = y - \mu(x^5 - vx^3 + x), \\
\dot{y} = -x,
\end{cases}
\quad \text{where } \mu \neq 0.
\]

If the parameter $v \geq \sqrt{5} \approx 2.2361$, then it has exactly two periodic solutions.

Each example indicates the strength of our theorems. In §5, we shall give more information and proofs of the above examples.
Periodic Solutions for Liénard Systems

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§ 2. Preliminary

In the section, we study basic properties of System (L) with (C1). Since $F, g$ are odd, the system is symmetric with respect to the origin, so we restrict our consideration on the region $y \geq F(x)$. Take a solution $\gamma$ which starts from the curve $y = F(x)$ at $x = -\rho < 0$ for itself at $x = \rho' \geq 0$. (The value $\rho'$ may possibly be zero or infinity.) We can represent the coordinate $y$ and the time $t$ as functions of $x$ as follows:

$$y = y(x), \quad t = t(x) \quad \text{on } (-\rho, \rho').$$

When we take $\hat{\gamma}$ as a solution, we denote its values by putting hats on them, for example, $\hat{\rho}$ in place of $\rho$. To simplify the proofs, we use the following notations:

$$y_1(x) = y(x), \quad y_2(x) = y(-x),$$

and

$$F_1(x) = F(x), \quad F_2(x) = F(-x).$$

Clearly, both $y_1, y_2$ are strictly decreasing. Since the system is symmetric, a solution $\gamma$ is periodic if and only if $\rho' = \rho$.

To study the solution, we introduce the following functions:

$$I(\rho) := -\int_{-\rho}^{\rho'} g(x)F(x)dt(x),$$

and

$$h(\rho) := -\int_{-\rho}^{\rho'} f(x)dt(x).$$

The first function has been used to judge whether a solution is periodic or not, and the second to judge whether a periodic solution is coarse or not. In the paper, we shall use the functions to know more accurate informations as follows.

Lemma 2.1. Consider System (L). Then the following hold:

(1) A solution $\hat{\gamma}$ is periodic if and only if $I(\hat{\rho}) = 0$. 
(2) A periodic solution \( \hat{\gamma} \) is outerly stable (unstable) if and only if \( I(\rho) < 0 \) (> 0) on \( (\hat{\rho}, \hat{\rho} + \epsilon) \) for some \( \epsilon > 0 \).

**Lemma 2.2.** Consider System (L). Then the following hold:

1. A periodic solution \( \hat{\gamma} \) is coarsely stable (unstable) if \( h(\hat{\rho}) < 0 \) (> 0).
2. A periodic solution \( \hat{\gamma} \) is outerly stable (unstable) if \( h(\rho) < 0 \) (> 0) on \( (\hat{\rho}, \hat{\rho} + \epsilon) \) for some \( \epsilon > 0 \).

Here we say that a solution is stable (unstable) if every near solution winds toward (backward) it, outerly stable (unstable) if every outerly near solution winds toward (backward) it.

**Proof of Lemma 2.1.** Clearly, the following formula holds:

\[
\frac{d}{dt} [ |G(x)| + (1/2)y^2 ] = -g(x)F(x).
\]

By integrating it, we obtain the following:

\[
I(\rho) = G(\rho') - G(\rho).
\]

Hence we can easily confirm the lemma.

**Proof of Lemma 2.2.** We shall prove only the second assertion. To simplify the proof, we transform the system into the following form:

\[
(L') \quad \begin{cases} \dot{x} = v, \\ \dot{v} = -g(x) - f(x)v, \end{cases} \quad \text{where } v := y - F(x).
\]

Now, we define a function \( P(\rho) := \rho' \) on a neighborhood of \( \hat{\rho} \), then by applying a known formula to it, see §6-2, we obtain the following:

\[
P'(\rho) = \frac{g(\rho)}{g(\rho')} \exp h(\rho) < \frac{g(\rho)}{g(\rho')} \quad \text{on } (\hat{\rho}, \hat{\rho} + \epsilon).
\]

By integrating it, we obtain the following:

\[
I(\rho) = G(\rho') - G(\rho) < 0 \quad \text{on } (\hat{\rho}, \hat{\rho} + \epsilon).
\]

Hence \( \hat{\gamma} \) is outerly stable.

**§3. Proof of Theorem A**

In the section, we shall prove Theorem A. It was already done by Alsholm, however, we shall do again because the methods we shall use here are also needed for proving Theorem B. For the simplicity, we add the following to (C2):
The function $F$ changes its sign plus to minus at odd-suffix zeros, and minus to plus at even-suffix zeros.

Of course, such an addition never hurts the generality.

**Lemma 3.1.** Consider System (L) with (C1), and a solution $\gamma$ with $\rho, \rho' \geq b$. Then the following hold:

1. When $F < 0$ on $(a, b)$, if $y_2(a) \leq y_1(a)$, then $y_2(b) < y_1(b)$.
2. When $F > 0$ on $(a, b)$, if $y_2(a) \geq y_1(a)$, then $y_2(b) > y_1(b)$.

**Proof.** We shall prove only the first assertion. To do it, we calculate as follows:

$$
\frac{d}{dx}[y_1 - y_2] = -\frac{g}{y_1 - F_1} + \frac{g}{y_2 - F_2} = \frac{g[y_1 - y_2]}{[y_1 - F_1][y_2 - F_2]}.
$$

It is equivalent to the following:

$$
\frac{d}{dx}[(y_1 - y_2) \exp\left(-\int_{a}^{b} \frac{g}{[y_1 - F_1][y_2 - F_2]} dx\right)] > 0.
$$

By integrating it, we obtain the following:

$$
y_1(b) - y_2(b) > y_1(a) - y_2(a) \geq 0.
$$

Hence we finish proving. \(\square\)

**Lemma 3.2.** Consider System (L) with (C1)–(C2), and a solution $\gamma$ with $\rho, \rho' \geq \delta_{k+1}$. Then the following hold:

1. When $k$ is odd, if $y_2(\xi) \leq y_1(\xi)$, $\beta_{k-1} \leq \xi \leq \beta_k$, then $y_2(\delta_{k+1}) < y_1(\delta_{k+1})$.
2. When $k$ is even, if $y_2(\xi) \geq y_1(\xi)$, $\beta_{k-1} \leq \xi \leq \beta_k$, then $y_2(\delta_{k+1}) > y_1(\delta_{k+1})$.

**Proof.** We shall prove only the first assertion. To do it by a contradiction, we assume the following:

1. $y_2(\xi) \leq y_1(\xi)$,
2. $y_2(\delta_{k+1}) \geq y_1(\delta_{k+1})$.

For it, we shall show the following:

1. $y_1(\phi(x)) < y_2(x)$,
2. $y_2(\phi(x)) < y_1(x)$ on $[\xi, \beta_k]$.

From now, we abbreviate the suffix of $\phi$. Firstly, we verify (2a). By applying Lemma 3.1 to (1b), we obtain the following:

$$
y_1(\phi(x)) \leq y_2(\phi(x)) < y_2(x) \quad \text{on} \quad [\xi, \beta_k].
$$

Thus (2a) is confirmed.
Secondly, we verify (2b). For it, we calculate as follows:
\[
\frac{d}{dx}[y_1 - y_2(\phi)] = -\frac{g}{y_1 - F_1} + \frac{g(\phi)\phi'}{y_2(\phi) - F_2(\phi)} > \frac{g[y_1 - y_2(\phi)]}{[y_1 - F_1][y_2(\phi) - F_2(\phi)]}.
\]
It is equivalent to the following:
\[
\frac{d}{dx}\left((y_1 - y_2(\phi)) \exp\left(-\int_{\xi}^{x} \frac{g}{y_1 - F_1}[y_2(\phi) - F_2(\phi)] dx\right)\right) > 0.
\]
By integrating it, we obtain the following:
\[
y_1(x) - y_2(\phi(x)) > y_1(\xi) - y_2(\phi(\xi)) \geq 0 \quad \text{on} \ [\xi, \beta_k].
\]
Thus (2b) is confirmed.

By using (2) and (C2), we can calculate as follows:
\[
(1/2)[y_1(\delta_{k+1})^2 - y_2(\delta_{k+1})^2] - (1/2)[y_1(\xi)^2 - y_2(\xi)^2] = -\int_{\xi}^{\beta_k} \frac{gF_1}{y_1 - F_1} dx - \int_{\beta_k}^{\beta_{k+1}} \frac{gF_1}{y_1 - F_1} dx + \int_{\xi}^{\beta_k} \frac{gF_2}{y_2 - F_2} dx + \int_{\beta_k}^{\beta_{k+1}} \frac{gF_2}{y_2 - F_2} dx \geq -\int_{\xi}^{\beta_k} \frac{gF_1}{y_1 - F_1} dx - \int_{\xi}^{\beta_k} \frac{g(\phi)\phi'F_1(\phi)}{y_1(\phi) - F_1(\phi)} dx + \int_{\xi}^{\beta_k} \frac{gF_2}{y_2 - F_2} dx + \int_{\xi}^{\beta_k} \frac{g(\phi)\phi'F_2(\phi)}{y_2(\phi) - F_2(\phi)} dx > -\int_{\xi}^{\beta_k} \frac{gF_1}{y_1 - F_1} dx + \int_{\xi}^{\beta_k} \frac{gF_2}{y_2 - F_2} dx + \int_{\xi}^{\beta_k} \frac{gF_1(\phi)}{y_2 - F_1(\phi)} dx + \int_{\xi}^{\beta_k} \frac{gF_2(\phi)}{y_1 - F_2(\phi)} dx = +\int_{\xi}^{\beta_k} \frac{gF_1[F_2(\phi) - F_1]}{[y_1 - F_1][y_2 - F_2(\phi)]} dx + \int_{\xi}^{\beta_k} \frac{gF_2[-F_1(\phi) + F_2]}{[y_2 - F_2][y_2 - F_1(\phi)]} dx \geq 0.
\]
It contradicts to (1). Hence we finish proving. \(\square\)

**Lemma 3.3.** Consider System (L) with (C1)–(C2), and a solution \(\gamma\) with \(\rho, \rho' \geq \beta_{k+1}\). Then the following hold:
1. When \(k\) is odd, if \(y_2(\beta_{k-1}) \leq y_1(\beta_{k-1})\), then \(y_2(\beta_{k+1}) < y_1(\beta_{k+1})\).
2. When \(k\) is even, if \(y_2(\beta_{k-1}) \geq y_1(\beta_{k-1})\), then \(y_2(\beta_{k+1}) > y_1(\beta_{k+1})\).
The above lemma is an immediate consequence of Lemmas 3.1 and 3.2. Now we have prepared all lemmas, we shall give a proof of Theorem A.

Proof of Theorem A. Consider a solution $\gamma$ with $\delta_{k+1} \leq \rho \leq \beta_{k+1}$, or $0 < \rho \leq \beta_1$ if $k = 0$. For it, we shall show the following:

\begin{align*}
(3a) \quad & \rho' > \rho \quad \text{for each odd } 1 \leq k \leq N . \\
(3b) \quad & \rho' < \rho \quad \text{for each even } 0 \leq k \leq N .
\end{align*}

We verify only the first. To do it by a contradiction, we assume that there is a solution $\hat{\gamma}$ with $\hat{\rho} \geq \rho = \rho'$, that is:

\begin{equation}
(4) \quad \hat{\gamma}_2(\rho) \geq F_2(\rho) \geq 0 \geq F_1(\rho) = \hat{\gamma}_1(\rho).
\end{equation}

By the definition, we obtain that $\hat{\gamma}_2(0) = \hat{\gamma}_1(0)$. By applying Lemmas 3.1, 3.2, and 3.3 to it, we obtain that $\hat{\gamma}_2(\rho) < \hat{\gamma}_1(\rho)$. It contradicts to (4). Thus (3a) is confirmed.

By using (3), we obtain the following:

\begin{align*}
(5a) \quad & I(\beta_k) < 0 , \quad I(\delta_{k+1}) > 0 \quad \text{for each odd } 1 \leq k \leq N . \\
(5b) \quad & I(\beta_k) > 0 , \quad I(\delta_{k+1}) < 0 \quad \text{for each even } 2 \leq k \leq N .
\end{align*}

Hence, in each interval $(\beta_k, \delta_{k+1})$, the function $I$ has a zero $\rho$, namely, the amplitude of a periodic solution. \hfill \Box

Just now we have proved Theorem A, however, we need one more knowledge about it to prove Theorem B.

Corollary 3.4. Under Theorem A, consider the outermost periodic solution $\hat{\gamma}$ with $\hat{\rho} = \hat{\rho}' < \delta_k$. Then the following hold:

1. If $k$ is odd, then $\hat{\gamma}$ is outerly stable.
2. If $k$ is even, then $\hat{\gamma}$ is outerly unstable.

Proof. We shall prove only the first assertion. Clearly, the amplitude $\hat{\rho}$ is the largest zero of the function $I$ on $(\beta_{k-1}, \delta_k)$. By using (5b), we obtain the following:

$$I(\rho) < 0 \quad \text{on } (\hat{\rho}, \delta_k].$$

Hence $\hat{\gamma}$ is outerly stable. \hfill \Box

§ 4. Proof of Theorem B

In the section, we shall prove Theorem B. The following lemma was given by Ding [Di] to prove his theorem, the idea is traced back to Zeng, see [ZZG].
Lemma 4.1. Consider System (L), and a solution $\gamma$ with $\rho$, $\rho' > b$. Then the following formula holds:

\[
- \int_{a}^{b} \frac{f}{y(x) - F(x)} \, dx = \log \left| \frac{y(a) - F(b)}{y(a) - F(a)} \right| - \int_{a}^{b} \frac{g(x)[F(b) - F(x)]}{[y(x) - F(b)][y(x) - F(x)]^2} \, dx.
\]

The formula is a key to prove Theorem B. By using it, we shall prepare two more lemmas to prove the theorem.

Lemma 4.2. Consider System (L) with (C1)–(C3), and a solution $\gamma$ with $\beta_{k-1} < \rho$, $\rho' \leq \alpha_k$. Then the following hold:

1. If $k$ is odd, then $h(\rho) < 0$.
2. If $k$ is even, then $h(\rho) > 0$.

Proof. We shall prove only the first assertion. By the definition, we obtain that $y_2(0) = y_1(0)$. By applying Lemmas 3.1 and 3.3 to it, we obtain the following:

\[(7a) \quad y_2(\beta_i) > y_1(\beta_i) \quad \text{for each odd } 1 \leq i \leq k - 1.
\]

\[(7b) \quad y_2(\beta_i) < y_1(\beta_i) \quad \text{for each even } 2 \leq i \leq k - 1.
\]

For each odd $1 \leq i \leq k - 1$, we shall show the following:

\[(8a) \quad y_1(\phi(x)) < y_2(\phi),
\]

\[(8b) \quad y_2(\phi) < y_1(x) \quad \text{on } [\beta_{i-1}, \beta_i].
\]

Firstly, we verify (8a). By applying Lemma 3.1 to (7a), we obtain the following:

\[y_1(\phi(x)) < y_1(x) < y_2(x) \quad \text{on } [\beta_{i-1}, \beta_i].
\]

Thus (8a) is confirmed.

Secondly, we verify (8b). For it, we calculate as follows:

\[
\frac{d}{dx}[y_1 - y_2(\phi)] = -\frac{g}{y_1 - F_1} + \frac{g(\phi)\phi'}{y_2(\phi) - F_2(\phi)} > \frac{g[y_1 - y_2(\phi)]}{[y_1 - F_1][y_2(\phi) - F_2(\phi)]}.
\]

It is equivalent to the following:

\[
\frac{d}{dx} \left[ (y_1 - y_2(\phi)) \exp \left( -\int_{\beta_{i-1}}^{x} \frac{g}{y_1 - F_1} \, dx \right) \right] > 0.
\]

By integrating it, we obtain the following:

\[y_1(x) - y_2(\phi(x)) > y_1(\beta_{i-1}) - y_2(\phi(\beta_{i-1})) \geq 0 \quad \text{on } [\beta_{i-1}, \beta_i].
\]

Thus (8b) is confirmed.
By using (C3), we obtain the following:

\begin{equation}
- \int_{\beta_{k-1}}^{\rho} f(x)dt(x) - \int_{-\rho}^{-\beta_{k-1}} f(x)dt(x) < 0.
\end{equation}

In addition, we shall show the following:

\begin{equation}
- \int_{\beta_{i+1}}^{\beta_{i-1}} f(x)dt(x) - \int_{-\beta_{i+1}}^{-\beta_{i-1}} f(x)dt(x) < 0 \quad \text{for each odd } 1 \leq i \leq k - 2.
\end{equation}

By using (6), (8a, b) and (C2), we calculate as follows:

\begin{align*}
- \int_{\beta_{i+1}}^{\beta_{i-1}} f(x)dt(x) + \int_{-\beta_{i+1}}^{-\beta_{i-1}} f(x)dt(x) \\
= - \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{f}{y-F} dx + \int_{-\beta_{i+1}}^{-\beta_{i-1}} \frac{f}{y-F} dx \\
= \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{gF_1}{y_1[y_1-F_1]^2} dx + \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{gF_1}{y_1[y_1-F_1]^2} dx \\
\quad - \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{gF_2}{y_2[y_2-F_2]^2} dx - \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{gF_2}{y_2[y_2-F_2]^2} dx \\
\leq \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{gF_1}{y_1[y_1-F_1]^2} dx + \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{g(\phi)\phi'F_1(\phi)}{y_1[\phi(y_1) - F_1(\phi)]^2} dx \\
\quad - \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{gF_2}{y_2[y_2-F_2]^2} dx - \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{g(\phi)\phi'F_2(\phi)}{y_2[\phi(y_2) - F_2(\phi)]^2} dx \\
< \int_{\beta_{i+1}}^{\beta_{i-1}} \left[ \frac{gF_1}{y_1[y_1-F_1]^2} + \frac{-gF_2}{y_1[y_1-F_2(\phi)]^2} \right] dx \\
\quad - \int_{\beta_{i+1}}^{\beta_{i-1}} \left[ \frac{-gF_2}{y_2[y_2-F_2]^2} dx + \frac{gF_1(\phi)}{y_2[y_2-F_1(\phi)]^2} \right] dx \\
= \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{g[-F_2(\phi) + F_1][y_1^2 - F_1F_2(\phi)]}{y_1[y_1-F_1]^2[y_1-F_2(\phi)]^2} dx \\
\quad + \int_{\beta_{i+1}}^{\beta_{i-1}} \frac{g[F_1(\phi) - F_2][y_2^2 - F_2F_1(\phi)]}{y_2[y_2-F_2]^2[y_2-F_1(\phi)]^2} dx \leq 0.
\end{align*}

Thus (10) is confirmed. By combining (9), (10), we obtain that \( h(\rho) < 0 \).

Hence we finish proving. \( \Box \)

**Lemma 4.3.** Consider System (L) with (C1)–(C4), and a solution \( \gamma \) with \( \alpha_k < \rho, \rho' < \delta_k \). Then the following hold:
(1) If \( k \) is odd, then \( h(\rho) \) is strictly increasing.
(2) If \( k \) is even, then \( h(\rho) \) is strictly decreasing.

Proof. We shall prove only the first assertion. To do it, consider two solutions \( \gamma, \hat{\gamma} \) with \( \alpha_k < \rho < \hat{\rho} < \delta_k, \alpha_k < \rho' < \hat{\rho}' < \delta_k \). For them, we shall show the following:

\[
\begin{align*}
(11a) & \quad - \int_0^{\alpha_k} f(x)d\hat{t}(x) > - \int_0^{\alpha_k} f(x)dt(x), \\
(11b) & \quad - \int_{-\alpha_k}^0 f(x)d\hat{t}(x) > - \int_{-\alpha_k}^0 f(x)dt(x).
\end{align*}
\]

We verify only the first. By using (C2), we obtain the following:

\[
F_1(\alpha_k) \geq F_1(x) \quad \text{on } [0, \alpha_k].
\]

By using (6), (12), we calculate as follows:

\[
- \int_0^{\alpha_k} f(x)d\hat{t}(x) + \int_0^{\alpha_k} f(x)dt(x) = \log \left| 1 - \frac{F_1(\alpha_k)}{\hat{y}_1(0)} \right| - \int_0^{\alpha_k} \frac{g[F_1(\alpha_k) - F_1]}{\hat{y}_1 - F_1} \cdot \frac{F_1(\alpha_k) - F_1}{F_1(\hat{\rho}) - F_1(\alpha_k)} \cdot \frac{F_1(\rho) - F_1(\alpha_k)}{F_1(\hat{\rho}) - F_1(\alpha_k)} dx > 0.
\]

Thus (11a) is confirmed.

To complete the proof, we shall show the following:

\[
\begin{align*}
(13a) & \quad - \int_{\alpha_k}^{\rho} f(x)d\hat{t}(x) > - \int_{\alpha_k}^{\rho} f(x)dt(x), \\
(13b) & \quad - \int_{-\rho}^{-\alpha_k} f(x)d\hat{t}(x) > - \int_{-\rho}^{-\alpha_k} f(x)dt(x).
\end{align*}
\]

We verify only the first. To do it, we define a function \( \eta_1 \) on \([\alpha_k, \rho]\) as follows:

\[
\eta_1(\xi) - F_1(\xi) = F_1(\xi) - F_1(\alpha_k) = \frac{F_1(\rho) - F_1(\alpha_k)}{F_1(\hat{\rho}) - F_1(\alpha_k)}.
\]

By (C4), we can use the following property, see §6-3:

(C4') On each interval \([\alpha_k, \delta_k], 2 \leq k \leq N + 1\), the following function is weakly monotone:

\[
D_k(x) := \frac{[F(x) - F(\alpha_k)]f(x)}{g(x)}.
\]
So we obtain the following:

\[
\frac{d\eta_1}{d\xi} = \frac{-g(\xi)}{\eta_1(\xi) - F_1(\xi)} \left[ \frac{D_k(\xi)}{D_k(x)} \right]
\]

\[
\geq \frac{-g(\xi)}{\eta_1(\xi) - F_1(\xi)} \quad \text{with } \eta_1(\rho) = y_1(\rho).
\]

By Comparison Theorem, we obtain the following:

\[
\eta_1(x) \leq y_1(x) \quad \text{on } [a_k, \rho].
\]

By using it, we can calculate as follows:

\[
-\int_{a_k}^{\rho} \frac{f}{\eta_1 - F_1} \, dx + \int_{a_k}^{\rho} \frac{f}{y_1 - F_1} \, dx
\]

\[
= -\int_{a_k}^{\rho} \frac{f}{\eta_1 - F_1} \, d\xi + \int_{a_k}^{\rho} \frac{f}{y_1 - F_1} \, dx = -\int_{a_k}^{\rho} \frac{[y_1 - \eta_1]f}{\eta_1 - F_1} \left[ y_1 - F_1 \right] \, dx > 0.
\]

Thus (13a) is confirmed. By combining (11), (13), we obtain that \( h(\dot{\rho}) > h(\rho) \). Hence we finish proving. \( \square \)

Until now, the bifurcation method has played an important role for estimating the number of limit cycles, see [Di], [LMP], [Ry2], [Zh1], [Zh2]. However, we shall carry out the proof of Theorem B without using it.

**Proof of Theorem B.** Consider the outermost periodic solution \( \dot{\gamma} \) with \( \dot{\rho} = \dot{\rho}' < \delta_k \). For it, we shall show the following:

\[
(14a) \quad h(\dot{\rho}) < 0 \quad \text{for each odd } 3 \leq k \leq N + 1,
\]

\[
(14b) \quad h(\dot{\rho}) > 0 \quad \text{for each even } 2 \leq k \leq N + 1.
\]

We verify only the first. To do it by a contradiction, we assume that \( h(\dot{\rho}) \geq 0 \). Then, by applying Lemma 4.2, we obtain that \( \dot{\rho} > \alpha_k \). By applying Lemma 4.3, we obtain that \( h(\rho) > 0 \) on \( (\dot{\rho}, \dot{\rho} + \epsilon) \). So \( \dot{\gamma} \) is outerly unstable. It contradicts to Corollary 3.4. Thus (14a) is confirmed.

We shall prove the uniqueness for each odd \( 2 \leq k \leq N + 1 \). To prove it by a contradiction, we assume that there is a neighboring periodic solution \( \gamma \) with \( \beta_{k-1} < \rho < \dot{\rho} \). (Remark that any stable periodic solution is isolated.) Then, by applying Lemmas 4.2 or 4.3 to (14a), we obtain that \( h(\rho) < 0 \). It contradicts to (14a), because two stable periodic solutions can not neighbor. Hence we finish proving. \( \square \)
§ 5. Applications—Proofs of Examples

In the section, we shall apply our theorems to the examples stated in §1. Before doing it, we shall give a proof of Choice Lemma.

Proof of Choice Lemma. Consider a mapping $\phi_k$ for (C2). For the simplicity, we assume that $\alpha_{k+1}$ is in $\phi_k[\beta_{k-1}, \beta_k]$. Even if not so, the proof proceeds similarly. We shall show that the following mapping also satisfies (C2):

$$\psi_k(x) := G^{-1}(G(x) + c_k), \quad c_k := G(\alpha_{k+1}) - G(\phi_k^{-1}(\alpha_{k+1})).$$

Clearly, it satisfies the following differential equation:

$$g(\psi_k(x))\psi_k'(x) = g(x) \quad \text{with } \psi_k(\phi_k^{-1}(\alpha_{k+1})) = \alpha_{k+1}.$$  

By integrating the differential inequality of (C2), we obtain the following:

$$\phi_k(x) \leq \psi_k(x) \leq \alpha_{k+1} \quad \text{on } [\beta_{k-1}, \phi_k^{-1}(\alpha_{k+1})],$$

$$\phi_k(x) \geq \psi_k(x) \geq \alpha_{k+1} \quad \text{on } [\phi_k^{-1}(\alpha_{k+1}), \beta_k].$$

Since the function $|F(x)|$ is increasing on $[\beta_k, \alpha_{k+1}]$, and decreasing on $[\alpha_{k+1}, \delta_{k+1}]$, we can calculate as follows:

$$|F(\psi_k(x))| \geq |F(\phi_k(x))| \geq |F(x)| \quad \text{on } [\beta_{k-1}, \beta_k].$$

Hence $\psi_k$ is better than $\phi_k$. \qed

Firstly, we consider System (Ex1). When $v = 1$, the system was studied by many mathematicians. First, Rychkov [Ry1] claimed the existence of at least $N$ periodic solutions, and later Zhang [Zh1] proved the exactness. Recently, Hara extended the parameter $v$ to all positive reals, and posed some problems on it, see §7-1. Our statement is an answer to his problems.

Proof of Example 1. To prove the claim, restrict the system to the strip $|x|^v \leq (N + 1)\pi$. Clearly, the function $F$ and its derivative $f$ respectively have $N + 1$ many positive zeros, namely,

$$\beta_k = (k\pi)^{1/v}, \quad \alpha_k = [(k - 1/2)\pi]^{1/v}, \quad 1 \leq k \leq N + 1.$$  

To apply our theorems, we choose the following mappings as the candidates for (C2):

$$\phi_k(x) = \sqrt{x^2 + c_k}, \quad c_k := [(k + 1/2)^{2/v} - (k - 1/2)^{2/v}]\pi^{2/v}.$$  

When $v < 1$, the function $F$ is not differentiable at $x = 0$, however, such a danger never hurts our theorems, see §6-4. We verify only (C4), that is, the
following function has a constant sign:

\[ f'(x) = \mu \nu^2 x^{v-2} \cos x^v (-x^v \tan x^v - 1/v + 1) \quad \text{on} \ [\alpha_{k+1}, \delta_{k+1}], \]

where \( \delta_{k+1} := [k^{2/v} + (k + 1/2)^{2/v} - (k - 1/2)^{2/v}]^{1/2} \pi^{1/v} \).

We verify it only when \( k \) is odd. By using Mean Value Theorem twice, we calculate as follows:

\[
\delta_{k+1}^v - (k + 1/2) \pi = \pi [k^{\lambda} + (k + 1/2)^{\lambda} - (k - 1/2)^{\lambda}]^{1/\lambda} - \pi [(k + 1/2)^{\lambda}]^{1/\lambda} \\
\leq (\pi/\lambda) [(k + 1/2)^{\lambda}]^{1/\lambda - 1} [k^{\lambda} - (k - 1/2)^{\lambda}] \\
= (\pi/2) \left( \frac{k}{k + 1/2} \right)^{\lambda - 1},
\]

where \( \lambda := 2/v \). By using it, we can calculate as follows:

\[
-x^v \tan x^v - 1/v + 1 \geq -\alpha_{k+1}^v \tan \delta_{k+1}^v - 1/v + 1 \\
\geq (k + 1/2) \pi \cot \left( \frac{\pi}{2} \left( \frac{k}{k + 1/2} \right)^{\lambda - 1} \right) - \lambda/2 + 1 \\
= \pi r \cot \left[ (\pi/2)(1 - 1/2r)^{\lambda - 1} \right] - \lambda/2 + 1,
\]

where \( r := (k + 1/2) \). Put it \( \chi(\lambda) \), then we obtain the following:

\[
\chi'(\lambda) = -\frac{(\pi^2 r/2)(1 - 1/2r)^{\lambda - 1} \log (1 - 1/2r)}{\sin^2 \left[ (\pi/2)(1 - 1/2r)^{\lambda - 1} \right]} - 1/2 \\
> -\pi r \log (1 - 1/2r) - 1/2 > \pi/2 - 1/2 > 0.
\]

So we obtain that \( \chi(\lambda) > 0 \) on \((1, \infty)\), and therefore, (C4) is confirmed. Hence the system has exactly \( N \) periodic solutions. \( \square \)

Secondly, we consider System (Ex2), which is the most famous system with a limit cycle. By applying his own theorems, Rychkov [Ry1] might know that the amplitude can be estimated by \( \sqrt{3} \approx 3.4641 \), and later Alsholm [Al] made the upper bound smaller to \( 2.8025 \). Our statement makes it better.

**Proof of Example 2.** Clearly, the polynomial \( F \) has one positive real root, namely, \( \beta = \sqrt{3} \). To apply our theorems, we choose the following mapping as the candidate for (C2):

\[ \phi(x) = \sqrt{x^2 + 2\sqrt{3}}, \quad \phi: [0, \beta] \to [\beta, \infty). \]
We verify only (C2) by calculating as follows:

\[
F(\phi(x))^2 - F(x)^2 = (2/3)\sqrt{3} \mu^2 \left[ x^4 - (4 - 2\sqrt{3})x^2 + (7 - 4\sqrt{3}) \right] \\
= (2/3)\sqrt{3} \mu^2 \left[ x^2 - (2 - \sqrt{3}) \right]^2 \geq 0 .
\]

So (C2) is confirmed. Hence the system has a unique periodic solution, whose amplitude \( \rho \) is estimated by \( \sqrt{3} \approx 1.7321 < \rho < \sqrt{3 + 2\sqrt{3}} \approx 2.5421 \). □

Thirdly, we consider System (Ex3), which is the simplest Liénard system with two limit cycles. First, Rychkov [Ry2] claimed the existence of exactly two limit cycles when the parameter \( v \geq 2.5 \), and later Alsholm [Al] made the lower bound smaller to 2.3178. Our statement makes it not only better but also the best possible in a sense, see §7-3.

**Proof of Example 3.** Clearly, the polynomial \( F \) has two positive real roots, namely,

\[
\beta_1, \beta_2 = \sqrt{\frac{v \pm c}{2}} , \quad c := \sqrt{v^2 - 4} .
\]

To apply our theorems, we choose the following mappings as the candidates for (C2):

\[
\phi_1(x) = \sqrt{x^2 + c} , \quad \phi_1: [0, \beta_1] \rightarrow [\beta_1, \beta_2] , \\
\phi_2(x) = \sqrt{x^2 + c} , \quad \phi_2: [\beta_1, \beta_2] \rightarrow [\beta_2, \infty) .
\]

We verify only (C2) by calculating as follows:

\[
F(\phi(x))^2 - F(x)^2 \\
= c\mu^2 \left[ 5x^8 - (8v - 10c)x^6 + (13v^2 - 12vc - 34)x^4 \\
\quad - (8v^3 + 8v^2c + 28v - 14c)x^2 + (2v^4 - 2v^3c - 10v^2 + 6vc + 9) \right] \\
= c\mu^2 (x^2 - \beta_1^2)^2 \left[ 5 \left( x^2 - \frac{3v - 5c}{2} \right)^2 + 16(v^2 - 5) \right] \geq 0 .
\]

So (C2) is confirmed. Hence the system has exactly two periodic solutions. □

§ 6. Remarks on theorems and proofs

1. By changing the coordinate \( x \) and the time \( t \), System (L) can be transformed into the following form, see [HY], for example:

\[
\begin{cases}
\dot{z} = y - F(x(z)) , \\
\dot{y} = -z ,
\end{cases}
\]

where \( x(z) := G^{-1}(z^2/2 \text{ sgn } z) \).
So, even when the functions $F, g$ are not necessarily odd, we can prove an extension of Theorem A in a similar way. The statement is as follows.

**Theorem A'.** Suppose that System (L) satisfies the following conditions:

(C1) The functions $F, g \in C^0(R)$ are continuous, $F(0) = g(0) = 0$, and $\text{sgn} \, g(x) = \text{sgn} \, x$.

(C2) The function $F_1 - F_2 := F(x(z)) - F(x(-z))$ has $N$ many positive zeros

$$ (\bar{\beta}_0 := 0 < \bar{\beta}_1 < \bar{\beta}_2 < \cdots < \bar{\beta}_N \quad (\bar{B} =: \bar{\beta}_{N+1}) $$

at which it changes its sign alternately, and for each $1 \leq k \leq N$, there are two $C^1$ mappings $\tilde{\phi}_{\pm k}: [\bar{\beta}_{k-1}, \bar{\beta}_k] \rightarrow [\bar{\bar{\beta}}_k, \bar{\bar{\beta}}_{k+1}]$, not necessarily onto, such that

$$ \bar{\phi}_{\pm k}(z) \bar{\phi}_{\pm k}(z) \geq z, $$

and either

$$ 0 \leq F_2(\bar{\phi}_{-k}(z)) \geq F_1(z) \geq F_2(z) \geq F_1(\bar{\phi}_k(z)) \leq 0 $$

or

$$ 0 \geq F_2(\bar{\phi}_{-k}(z)) \leq F_1(z) \leq F_2(z) \leq F_1(\bar{\phi}_k(z)) \geq 0. $$

Then it has at least $N$ periodic solutions, all of whose amplitudes $\rho, \rho'$ are estimated by $\tilde{\beta}_k < -x^{-1}(-\rho), x^{-1}(\rho') < \bar{\tilde{\phi}}_k(\tilde{\beta}_k)$ for some $1 \leq k \leq N$.

The second conclusion of Theorem A' asserts that the system has no periodic solutions in the strip $x(-\tilde{\beta}_1) \leq x \leq x(\tilde{\beta}_1)$. It is the typical theorem assuring no periodic solutions, see [SH].

2. In the proof of Lemma 2.2, we use a minor formula. To explain it, consider a planar vector field $X$ and a solution $\gamma = \{p(s)\}$, where the parameter $s$ is so taken that $|p'(s)| = 1$, where $| \cdot |$ denotes the length of vector. On a neighborhood of $\gamma$, we take new coordinates $(s, n)$ as follows:

$$ (x, y) = p(s) + nX^\perp(p(s)), $$

where $X^\perp$ is a unit orthogonal vector field of $X$. Under the coordinates, consider a solution $n = n(s, n_0)$ starting from $(0, n_0)$, then the following formula holds:

$$ \frac{\partial n}{\partial n_0}(l, 0) = \frac{|X(p(0))|}{|X(p(l))|} \exp \left[ \int_0^l \text{div} \, X \, dt(s) \right]. $$

It is not formulated in any texts, however, an idea of the proof is found in [Zh3, IV, §2] or [Ye, §2].
3. After the first version of the manuscript had been written, Professor Zhang informed the author that (C4) can be weakened to (C4'), which is formulated in the proof of Lemma 4.3. Thanks to her advice, we can take her method [Zh2] in our proof, the idea is traced back to Zeng, see [ZZG]. We can formulate the weakness of (C4') as follows.

**Lemma.** If System (L) satisfies (C1)–(C4), then (C4') too.

**Proof.** We prove it only when $k$ is odd. To do it, we assume that the derivative $f$ is weakly decreasing. Then we calculate as follows:

\[
\frac{d}{dx} \frac{F(x) - F(x_k)}{x - x_k} = \frac{f(x)(x - x_k) - [F(x) - F(x_k)]}{(x - x_k)^2} = \frac{f(x) - f(c)}{x - x_k} \leq 0,
\]

where $x_k < c < x$. Hence the function $D_k(x)$ is weakly decreasing. \(\square\)

4. Theorem B requires System (L) to be of class $C^1$, however, when $v < 1$, System (Ex1) is not differentiable on $x = 0$. We can avoid it as follows. By changing the coordinate $x$ and the time $t$, it can be transformed into the following system:

\[
\begin{aligned}
\dot{z} &= y - \mu \sin z, \\
\dot{y} &= -(\gamma/2)|z|^{\lambda-1} \text{sgn } z,
\end{aligned}
\]

where $\lambda := 2/v > 2$.

Clearly, it is of class $C^1$, so we can apply Lemma 2.2 to it. Other parts of the proof are not problem, so we can apply Theorem B to it.

§ 7. Comments on examples

1. In a lecture given at Beijing Branch of Academia Sinica, T. Hara posed some conjectures on System (Ex1). The following are parts of his conjectures:

(1) The number of periodic solutions is a weakly decreasing function of $\mu > 0$.

(2) The number of periodic solutions is a weakly decreasing function of $v > 0$.

Many results of his computer experiments indicate the validity of the above conjecture.

2. Methods which assume several periodic solutions are also effective for estimating the amplitude $\rho = \rho(\mu)$ of the limit cycle of System (Ex2). For example, by using a theorem of Neumann and Sabbagh [NS], when the parameter $\mu \leq 0.4$, it can be estimated by $\rho(\mu) < \sqrt{6} \approx 2.4495$. 

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On the other hand, by using a result of K.-C. Huang, see [Zh3] or [Ye], it can be estimated by \( \rho(\mu) < B(\mu) \), where \( B(\mu) \) is the largest root of the algebraic equation
\[
B^3 - 3(1 + 1/\mu)B - 2 = 0.
\]
(If we use a theorem of Lloyd [L1], we can make a more accurate estimation.) The more we take \( \mu \) large, the more effective this estimation becomes, in fact, the upper bound \( B(\mu) \to 2 \) as \( \mu \to \infty \). (It is already known that the amplitude \( \rho(\mu) \to 2 \) as \( \mu \to 0 \) or \( \mu \to \infty \), see [Le, VII].)

3. We shall remark the following fact, that is: Our results are not affected by constant multipliers of \( F \). It seems to be a defect of our theorems, because multipliers of \( F \) has a large influence. For example, System (Ex3) has the following properties:

(1) If the parameter \( v < \sqrt{5} \approx 2.2361 \), then it has no periodic solutions for sufficiently large \( \mu > 0 \).

(2) If the parameter \( v > (2/3)\sqrt{10} \approx 2.1082 \), then it has exactly two periodic solutions for sufficiently small \( \mu > 0 \).

The results are already mentioned in [Al], and can be proved by the large and the small parameter methods, see [Le, VII].

Note that (1.4) of [L1] is equivalent to the system, and the parameters are linked by \( 5(k+1)^2 = 9kv^2 \). So the table of [L1, p. 569] is also useful for studying the system.

**References**


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