Stability Region for Systems of Differential-Difference Equations

By

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Dedicated to Professor Roberto Conti and Professor Gaetano Villari on the occasion of their 70th birthday

Abstract. In this paper we give some new necessary and sufficient conditions under which the zero solution of the linear differential-difference system

\[ (L) \quad \dot{x}(t) = Ax(t - \tau) \]

is asymptotically stable where \( A \) is an \( n \times n \) matrix. Our stability criteria are expressed explicitly in terms of the trace, the determinant and the eigenvalues of the matrix \( A \). For example, in case

\[ A = -\rho \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (\rho > 0, -\pi < \theta \leq \pi), \]

the zero solution of \((L)\) is asymptotically stable if and only if

\[ 0 < \rho \tau < \frac{\pi}{2} - |\theta|. \]

The proof of our results is carried out by an application of Pontryagin criterion for quasi-polynomials to the characteristic equation of \((L)\). Some phase portraits of trajectories of differential-difference system drawn by a computer are also attached.

1. Introduction

In studying the stability of the zero solution of a functional differential equation with two parameters, main concern is on the stability region, the maximal region in the space of parameters for which the zero solution of a functional differential equation with two parameters is asymptotically stable. We can find some stability regions in several books [1–4, 6]. For example, consider a scalar differential-difference equation with two parameters

\[ (1.1) \quad \dot{x}(t) = -\alpha x(t - \tau) - \beta x(t) \]

where \( \tau > 0 \). Then it is known that the stability region for \((1.1)\) is the union

This work was partially supported by a grant for Basic Science Research of the Sumitomomo Foundation.
of the following regions
\[ \{(\alpha, \beta): \beta = -\alpha \cos \zeta, \ 0 < \alpha \tau \sin \zeta < \zeta, \ 0 < \zeta < \pi \} \]
and
\[ \{(\alpha, \beta): -\beta < \alpha \leq \beta \} . \]

The purpose of this paper is to determine the stability region in the parameter space \((\text{tr } A, \det A)\) for the system

\[ \dot{x}(t) = Ax(t - \tau) \tag{1.2} \]

where \(A\) is a \(2 \times 2\) matrix. It is well-known that the zero solution of (1.2) is asymptotically stable if and only if no zeros of characteristic function
\[ \det (zE - A \exp (-\tau z)) \]
lie in the half-plane \(\text{Re } z \geq 0\). However, in general, it is not so easy to verify whether or not this necessary and sufficient condition is satisfied for a specific matrix \(A\). The condition given provides an explicit expression for the zeros of the characteristic functions not to lie in the half plane \(\text{Re } z \geq 0\) (Theorem 3.1).

To obtain our result Theorem 2.1, in Section 2, we consider the quasi-polynomial
\[ A(z) = z^2 e^{2z} - (\text{tr } A)z e^z + \det A \]
which corresponds to the characteristic function of (1.2) with \(\tau = 1\) and present a necessary and sufficient condition for all zeros of \(A(z)\) to have negative real parts by means of the Pontryagin criterion [5].

In Theorem 3.1, using \(\text{tr } A\) and \(\det A\) as two parameters, we give the stability region for (1.2). For a special case
\[ A = -\rho \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \]
where \(\rho > 0\) and \(-\pi < \theta \leq \pi\), the stability region for (1.2) is given by

\[ \left\{ (\theta, \rho \tau): 0 < \rho \tau < \frac{\pi}{2} - |\theta| \right\} \tag{1.3} \]

(Theorem 3.3). In case \(\theta = 0\), namely, \(A\) is a diagonal matrix, this result yields the necessary and sufficient condition
\[ 0 < \alpha \tau < \frac{\pi}{2} \]
for the asymptotic stability of the zero solution of (1.1) with \(\beta = 0\). We also
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give an explicit expression for the zeros of the characteristic function of the $n$-dimensional system

$$\dot{x}(t) = Ax(t - \tau)$$

not to lie in the half plane $\Re z \geq 0$ (Theorem 3.4).

2. Zeros of characteristic quasi-polynomials

In [5] Pontryagin investigated the quasi-polynomial

$$\Delta(z) = \sum_{m=0}^{p} \sum_{n=0}^{q} a_{mn} z^{m} e^{nz}$$

and gave necessary and sufficient conditions for all the zeros of $\Delta(z)$ to have negative real parts, that is, analytical stability conditions. Applications to some specific quasi-polynomials are given by means of his criterion. One of those results can be found in [1–3]. For other criteria, see the recent books [4, 6]. It is important to note that $\Delta(z) = 0$ is the characteristic equation for a differential-difference equation with a single delay or with the delays whose ratios are rational.

In this section we will discuss the stability of $z^2 e^{2z} - (\text{tr } A) z e^{z} + |A|$, where $A$ is a $2 \times 2$ matrix. Let us examine, first, Pontryagin’s results. Consider quasi-polynomial $\Delta(z)$. The term $a_{rs} z^{r} e^{sz}$ is called the principal term of quasi-polynomial $\Delta(z)$ if it has the following property $(P)$:

(P) \hspace{1cm} a_{rs} \neq 0; \hspace{0.5cm} a_{mn} = 0 \hspace{0.5cm} \text{for } m > r \hspace{0.5cm} \text{or } n > s .

In case $a_{pq} \neq 0$, of course, $a_{pq} z^{p} e^{sz}$ is the principal term.

Theorem A [5]. Suppose that quasi-polynomial $\Delta(z)$ has a principal term. If all the zeros of $\Delta(z)$ lie to the left side of the imaginary axis, then the zeros of $F(y)$ and $G(y)$ are real, simple, alternating and

$$G'(y)F(y) - G(y)F'(y) > 0 \hspace{0.5cm} \text{for } y \in \mathbb{R} ,$$

where $\Delta(iy) = F(y) + iG(y)$, $i^2 = -1$. Conversely, all zeros of $\Delta(z)$ have negative real parts provided that one of the following conditions is satisfied:

(i) all the zeros of $F(y)$ and $G(y)$ are real, simple and alternating and inequality (2.1) is satisfied for at least one $y \in \mathbb{R} ;$

(ii) all the zeros of $F(y)$ or $G(y)$ are real, simple and alternating and inequality (2.1) is satisfied for each zeros.

Pontryagin also obtained a necessary and sufficient condition under which all the roots of $F(y)$ and/or $G(y)$ are real. Let us consider a polynomial of
the form
\[
    f(z, u, v) = \sum_{m=0}^{p} \sum_{n=0}^{q} b_{mn}z^m \phi^{(n)}(u, v),
\]
where \(\phi_m^{(n)}(u, v)\) is a homogeneous polynomial of degree \(n\) in \(u\) and \(v\). The principal term in the polynomial \(f(z, u, v)\) is the term \(b_{rs}z^r \phi_{r}^{(s)}(u, v)\) which possesses property \((P)\). Let \(b_{rs}z^r \phi_{r}^{(s)}(u, v)\) denote the principal term of \(f(z, u, v)\). Then
\[
    f(z, u, v) = z^r \phi_{r}^{(s)}(u, v) + \sum_{m=0}^{r-1} \sum_{n=0}^{s} b_{mn}z^m \phi_{m}^{(n)}(u, v)
\]
where \(\phi_{r}^{(s)}(u, v) = \sum_{n=0}^{s} b_{rn} \phi_{r}^{(n)}(u, v)\).

Theorem B [5]. Let \(f(z, u, v)\) be a polynomial with a principal term \(b_{rs}z^r \phi_{r}^{(s)}(u, v)\). If \(\epsilon\) is such that \(\Phi^{(s)}(\epsilon + iy) \neq 0\) for \(y \in \mathbb{R}\), then for sufficiently large integers \(k\), the function \(F(z) = f(z, \cos z, \sin z)\) will have exactly \(4sk + r\) zeros in the interval \(-2k\pi + \epsilon \leq \text{Re} z \leq 2k\pi + \epsilon\). Thus, the equation \(F(z) = 0\) will have only real roots if and only if in the interval \(-2k\pi + \epsilon \leq \text{Re} z \leq 2k\pi + \epsilon\) it has exactly \(4sk + r\) real roots for a sufficiently large \(k\).

Let us denote by \(\text{tr} A\) the trace of a matrix \(A\) and by \(|A|\) the determinant of \(A\). We now give our analytical stability conditions of the quasi-polynomial \(A(z) = z^2e^{2z} - (\text{tr} A)ze^z + |A|\) and show the stability region for \(A(z)\).

Theorem 2.1. Let \(A\) be a \(2 \times 2\) matrix of real constants. Then all zeros of the quasi-polynomial \(z^2e^{2z} - (\text{tr} A)ze^z + |A|\) have negative real parts if and only if
\[
    (\frac{\pi}{2})^2 + \frac{\pi}{2}\text{tr} A + |A| > 0
\]
and
\[
    0 < |A| < \zeta^2 < (\frac{\pi}{2})^2
\]
where \(\zeta\) is the smallest positive root of the equation \(y \sin y = -\frac{1}{2}(\text{tr} A)\).

Remark 2.1. The smallest positive root \(\zeta\) of \(y \sin y = -\frac{1}{2}(\text{tr} A)\) satisfies
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\[ 0 < \zeta < \frac{\pi}{2} \text{ if} \]

\begin{equation}
-\pi < \text{tr} \, A < 0
\end{equation}

(see Fig. 1). Conversely, if \( 0 < \zeta < \frac{\pi}{2} \), then we have

\[ 0 < \zeta \sin \zeta = -\frac{1}{2} (\text{tr} \, A) < \frac{\pi}{2}, \]

that is, (2.4) holds. Hence, (2.3) implies (2.4).

In Figure 2 we show the region which satisfies (2.2) and (2.3).

**Proof of Theorem 2.1.** Let \( A(z) = z^2 e^{2z} - (\text{tr} \, A) e^z + |A| \). Then we have

\[ A(iy) = F(y) + iG(y), \quad y \in \mathbb{R} \]

with

\[ F(y) = -y^2 \cos 2y + (\text{tr} \, A)y \sin y + |A| \]

and

\[ G(y) = -y^2 \sin 2y - (\text{tr} \, A)y \cos y. \]

If \( g(y, u, v) = -2uvy^2 - (\text{tr} \, A)uy \), then \( G(y) = g(y, \cos y, \sin y) \) and the function \( \Phi_*^{(2)}(z) \) in Theorem B is \( -\sin 2z \). Take \( \epsilon = \frac{\pi}{4} \). Then

\[ \varphi(y) = y \sin y. \]

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Fig. 1
\[ \Phi_{*}^{(2)}\left(\frac{\pi}{4} + iy\right) = -\cosh 2y < 0 \quad \text{for } y \in \mathbb{R}. \]

Hence, Theorem B implies the zeros of \( G(y) \) are real if and only if \( G(y) \) has exactly \( 8k + 2 \) roots in the interval \( \left[-2k\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{4}\right] \) for sufficiently large integers \( k \). The equation \( G(y) = 0 \) has roots at \( y = 0, y = \left(n + \frac{1}{2}\right)\pi \) (\( n \) is any integer) and the zeros of \( y \sin y = \frac{1}{2}(\text{tr } A) \) because

\[ G(y) = -y \cos y(2y \sin y + \text{tr } A). \]

**Sufficiency.** Assume (2.4). Then there are \( 2k + 1 \) positive roots and \( 2k \) negative roots of \( y \sin y = \frac{1}{2}(\text{tr } A) \) for \( -2k\pi + \frac{\pi}{4} \leq y \leq 2k\pi + \frac{\pi}{4} \) (see also Fig. 1), and therefore \( G(y) \) has exactly \( 8k + 2 \) zeros in this interval. Hence by Theorem B all the zeros of \( G(y) \) are real.
We will show that conditions (2.2)–(2.4) imply inequality (2.1) in Theorem A is satisfied at the zeros of \( G(y) \), that is, \( G'(y)F(y) \) is always positive at such zeros. Since

\[
G'(y) = -2y^2 \cos 2y - 2y \sin 2y + (\text{tr } A)y \sin y - (\text{tr } A) \cos y,
\]

we have \( G'(0)F(0) = -(\text{tr } A)|A| > 0 \), namely, (2.1). Let us check inequality (2.1) at the other zeros of \( G(y) \).

**Case 1:** \( \cos y = 0 \). Let \( y_n = \left( n + \frac{1}{2} \right) \pi, \ n \) integer. Then we get

\[
G'(y_n)F(y_n) = y_n \{ 2y_n + (\text{tr } A) \cos y_n \} \{ y_n^2 + (\text{tr } A)y_n \sin y_n + |A| \}.
\]

From (2.4) it follows that

\[
y_n(2y_n \pm \text{tr } A) > 0 \quad \text{and} \quad y_n^2 \pm (\text{tr } A)y_n \geq y_0^2 + (\text{tr } A)y_0
\]

for each \( n \). Hence, we have

\[
G'(y_n)F(y_n) = y_n(2y_n + (-1)^n \text{tr } A)\{ y_n^2 + (-1)^n(\text{tr } A)y_n + |A| \}
\]

\[
\geq y_n(2y_n + (-1)^n \text{tr } A)\{ y_0^2 + (\text{tr } A)y_0 + |A| \} > 0
\]

by (2.2).

**Case 2:** \( y \sin y = -\frac{1}{2}(\text{tr } A) \). Let \( \zeta_n \) be positive roots of \( y \sin y = -\frac{1}{2}(\text{tr } A) \)
such that \( y_{n-1} < \zeta_n < y_n \) with \( n \geq 0 \). Then it follows from (2.3) that \( \sqrt{|A|} < \zeta \leq \zeta_n \) for \( n \geq 0 \). A simple calculation yields

\[
G'(\zeta_n)F(\zeta_n) = \left\{ 2\zeta_n^2 - (\text{tr } A) \cos \zeta_n - \frac{1}{2}(\text{tr } A)^2 \right\}(\zeta_n^2 - |A|).
\]

Since \( \cos \zeta_n > 0 \) with \( n \) even, we see that \( G'(\zeta_n)F(\zeta_n) > 0 \) in case \( n \) is even. Suppose that \( n \) is odd. Then

\[
2\zeta_n^2 - (\text{tr } A) \cos \zeta_n \geq 2\zeta_1^2 - (\text{tr } A) \cos \zeta_1.
\]

In fact, we have

\[
2\zeta_n^2 - (\text{tr } A) \cos \zeta_n > 2\zeta_n^2 + \text{tr } A > 2\left( n - \frac{1}{2} \right)^2 \pi^2 - \pi
\]

\[
> 2\left( n - \frac{3}{2} \right)^2 \pi^2 > 2\zeta_{n-2}^2 > 2\zeta_{n-2}^2 - (\text{tr } A) \cos \zeta_{n-2}
\]

for \( n = 3, 5, \ldots \). Since \( y \sin y \geq \frac{\pi}{2} \) for \( y \in \left[ \frac{\pi}{2}, 3\pi \right] \) and \( \zeta_1 \sin \zeta_1 = -\frac{1}{2}(\text{tr } A) < \frac{\pi}{2} \),
\[ \frac{\pi}{2}, \text{ we see that } \zeta_{1} > \frac{2}{3}\pi. \text{ Hence,} \]

\[ 2\zeta_{1}^{2} - (\text{tr } A) \cos \zeta_{1} - \frac{1}{2}(\text{tr } A)^{2} > \frac{8}{9}\pi^{2} - \pi - \frac{1}{2}\pi^{2} > 0. \]

We therefore conclude that \( G'(-\zeta_{n})F(-\zeta_{n}) > 0 \) with \( n \) odd. Note that each \( (-\zeta_{n}) \) is also a root of \( y \sin y = \frac{1}{2}(\text{tr } A) \) and

\[ G'(-\zeta_{n})F(-\zeta_{n}) = G'(\zeta_{n})F(\zeta_{n}) > 0. \]

Thus, condition (2.1) is satisfied at all the zeros of \( G(y) \), and therefore, by Theorem A all zeros of \( \Delta(z) \) have negative real parts.

**Necessity.** Suppose that all zeros of \( \Delta(z) \) lie to the left of the imaginary axis. Then Theorem A asserts that the zeros of \( G(y) \) are real and simple, and that the function \( G(y)F(y) \) are positive at the zeros of \( G(y) \).

We must have \( \text{tr } A \neq 0 \), for otherwise, the equation \( G(y) = 0 \) has a triple root at \( y = 0 \), which is a contradiction. If \( \text{tr } A > 0 \), then there exist at most \( 2k \) positive roots and \( 2k - 1 \) negative roots of \( y \sin y = -\frac{1}{2}(\text{tr } A) \) in the interval \( \left[ -2k\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{4} \right] \) for \( k \) sufficiently large. Hence, \( G(y) \) has at most \( 8k \) zeros in this interval, which contradicts Theorem B. Suppose that the graph \( y \sin y \) has the first local maximum \( \mu > \frac{\pi}{2} \) at \( y = v \in \left( \frac{\pi}{2}, \pi \right) \). If \( \text{tr } A \leq -2\mu \), then the equation \( y \sin y = -\frac{1}{2}(\text{tr } A) \) has at most \( 2k - 1 \) positive roots and \( 2k - 2 \) negative roots for \( -2k\pi + \frac{\pi}{4} \leq y \leq 2k\pi + \frac{\pi}{4} \). Consequently, the number of zeros of \( G(y) \) is not greater than \( 8k - 2 \), which is also a contradiction. We then conclude that

\[ -2\mu < \text{tr } A < 0. \]

We claim that condition (2.2) is satisfied. Suppose that this assertion is false. Then

\[ \text{tr } A < -\pi \]

because

\[ G'(y_{0})F(y_{0}) = \frac{\pi}{2}(\pi + \text{tr } A)\left\{ \frac{1}{4}\pi^{2} + \frac{\pi}{2}(\text{tr } A) + |A| \right\} > 0. \]
It follows from (2.5) and (2.6) that

$-2\mu < \text{tr} A < -\pi$.

Take notice of the positions of the intersecting points of the curve $y \sin y$ and the horizontal line $-\frac{1}{2}(\text{tr} A)$. Let $\zeta$ be the smallest positive root of $y \sin y = -\frac{1}{2}(\text{tr} A)$. Then by (2.7) we obtain

$$\frac{\pi}{2} < \zeta < \nu \quad \text{and} \quad \frac{d}{dy}(y \sin y) \bigg|_{y=\zeta} > 0.$$ 

Hence we have

$$2\zeta^2 - (\text{tr} A) \cos \zeta - \frac{1}{2}(\text{tr} A)^2 = 2\zeta^2 + 2\zeta \sin \zeta \cos \zeta - 2\zeta^2 \sin^2 \zeta$$

$$= 2\zeta \cos \zeta \left(\zeta \cos \zeta + \sin \zeta\right)$$

$$= 2\zeta \cos \zeta \frac{d}{dy}(y \sin y) \bigg|_{y=\zeta} < 0.$$ 

Since

$$G'(\zeta)F(\zeta) = \left\{2\zeta^2 - (\text{tr} A) \cos \zeta - \frac{1}{2}(\text{tr} A)^2\right\}(\zeta^2 - |A|) > 0,$$

we get $\zeta^2 < |A|$. This inequality implies that

$$0 > \frac{1}{4}\pi^2 + \frac{\pi}{2}(\text{tr} A) + |A| > \frac{1}{4}\pi^2 + \frac{\pi}{2}(\text{tr} A) + \zeta^2$$

$$= \frac{1}{4}\pi^2 - \pi \zeta \sin \zeta + \zeta^2$$

$$> \left(\frac{\pi}{2} - \zeta\right)^2 > 0,$$

which is a contradiction. We therefore conclude that conditions (2.2) and (2.4) hold.

Next, we will show that condition (2.3) is necessary. Since $G'(0)F(0) = -(\text{tr} A)|A|$, we obtain $|A| > 0$. Note that condition (2.4) implies $0 < \zeta < \frac{\pi}{2}$, and so
cos ζ > 0 and \( \frac{d}{dy}(y \sin y) \bigg|_{y=\zeta} > 0 \).

Hence we must have

\[ |A| < \zeta^2 \]

because

\[
G'(\zeta)F(\zeta) = \left\{ 2\zeta^2 - (\text{tr } A) \cos \zeta - \frac{1}{2} (\text{tr } A)^2 \right\} (\zeta^2 - |A|)
\]

\[ = 2\zeta (\zeta^2 - |A|) \cos \frac{d}{dy}(y \sin y) \bigg|_{y=\zeta} > 0. \]

Therefore, condition (2.3) is satisfied. The proof is now complete.

In case \( A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \), we have the following result.

**Corollary 2.1.** All zeros of the quasi-polynomial \( z^2 e^{2z} + 2axe^z + \alpha^2 + \beta^2 \), where \( \alpha \) and \( \beta \) are real, have negative real parts if and only if

\[ 0 < \alpha < \frac{\pi}{2} \quad \text{and} \quad \alpha^2 + \beta^2 < \zeta^2 < \left( \frac{\pi}{2} \right)^2 \]

where \( \zeta \) is the smallest positive root of the equation \( y \sin y = \alpha \).

Figure 3 illustrates the region of the \( (\alpha, \beta) \)-plane in which all the zeros of \( z^2 e^{2z} + 2axe^z + \alpha^2 + \beta^2 \) have negative real parts.
3. Linear systems of differential-difference equations

Consider the autonomous linear system of the form

\[ \dot{x}(t) = Ax(t - \tau), \]

where \( A \) is a \( 2 \times 2 \) real constant matrix and \( \tau \) is a positive constant. It is well-known that for system (3.1) the asymptotic stability is equivalent to the uniform asymptotic stability and to the exponential asymptotic stability. Moreover, a necessary and sufficient condition of asymptotic stability of the zero solution of (3.1) is the absence of roots of the characteristic equation

\[ z^2 \exp(2\tau z) - (\text{tr} \, A)z \exp(\tau z) + |A| = 0 \]

for (3.1) in the half-plane \( \text{Re} \, z \geq 0 \). The detail can be found in [3, Chapters 6 and 7].

By using our results in Section 2, we give another expression of the necessary and sufficient condition, represented explicitly in terms of \( \text{tr} \, A, |A| \) and \( \tau \), for the zero solution of (3.1) to be asymptotically stable.

**Theorem 3.1.** The zero solution of (3.1) is asymptotically stable if and only if

\[ 2\sqrt{|A|} \sin \tau \sqrt{|A|} < -\text{tr} \, A < \frac{\pi}{2\tau} + \frac{2\tau |A|}{\pi} \]

and

\[ 0 < \tau^2 |A| < \left( \frac{\pi}{2} \right)^2. \]

**Proof.** By the change of variable \( u(t) = x(\tau t) \), system (3.1) is transformed to a system

\[ \dot{u}(t) = \tau Au(t - 1). \]

Note that the zero solution of (3.1) is asymptotically stable if and only if the zero solution of (3.1)' is asymptotically stable. The quasi-polynomial for system (3.1)' is given by

\[ z^2 e^{\tau z} - (\text{tr} (\tau A)) z e^{\tau z} + |\tau A|. \]

Hence by virtue of Theorem 2.1 we get that all zeros of the above quasi-polynomial have negative real parts if and only if

\[ \left( \frac{\pi}{2} \right)^2 + \frac{\pi}{2} (\text{tr} (\tau A)) + |\tau A| > 0 \]
and

\[(3.3)' \quad 0 < |\tau A| < \zeta^2 < \left(\frac{\pi}{2}\right)^2\]

where \(\zeta\) is the smallest positive root of the equation \(y \sin y = -\frac{1}{2} (\text{tr} \,(\tau A))\). It is easy to see that conditions (3.2)' and (3.3)' are equivalent to conditions (3.2) and (3.3). Thus, the conclusion can be deduced from Corollary 6.1 in [1].

Remark 3.1. Let \(\tau \to 0\). Then conditions (3.2) and (3.3) are reduced to

\[(3.4) \quad \text{tr} \, A < 0 \quad \text{and} \quad |A| > 0\]

which is the Routh-Hurwitz criterion for the characteristic equation of the ordinary differential equation \(\dot{x} = Ax\) with a \(2 \times 2\) constant matrix \(A\).

We will discuss further two special cases:

\[(3.5) \quad A = -\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix} \quad (\lambda_1, \lambda_2, b \in \mathbb{R});\]

\[(3.6) \quad A = -\rho \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (\rho > 0, -\pi < \theta \leq \pi).\]

Consider the former case. Then conditions (3.2) and (3.3) hold if and only if

\[(3.7) \quad 0 < \lambda_1 \tau < \frac{\pi}{2} \quad \text{and} \quad 0 < \lambda_2 \tau < \frac{\pi}{2},\]

since \(|A| = \lambda_1 \lambda_2\) and \(\text{tr} \, A = - (\lambda_1 + \lambda_2)\), and so the stability condition in Theorem 3.1 is not affected by \(b\).

We have thus proved the following:

**Theorem 3.2.** The zero solution of (3.1) with (3.5) is asymptotically stable if and only if condition (3.7) holds.

In the latter case we obtain the following result.

**Theorem 3.3.** The zero solution of (3.1) with (3.6) is asymptotically stable if and only if

\[(3.8) \quad 0 < \rho \tau < \frac{\pi}{2} - |\theta| .\]

**Proof.** It follows from Corollary 4.1 in [3, Chapter 7] and Corollary 2.1 that the zero solution of (3.1) with
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\[ A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \]

is asymptotically stable if and only if

\[(3.9) \quad 0 < \alpha \tau < \frac{\pi}{2} \quad \text{and} \quad (\alpha^2 + \beta^2)\tau^2 < \zeta^2 \]

where \( \zeta \) is the smallest root of the equation \( y \sin y = \alpha \tau, \ 0 < \zeta < \frac{\pi}{2} \).

Letting \( \alpha = \rho \cos \theta \) and \( \beta = \rho \sin \theta \), we will show conditions (3.8) and (3.9) are equivalent. Note that the function \( y \sin y \) is monotone increasing in the interval \([0, \frac{\pi}{2}]\).

If condition (3.9) is satisfied, then we have

\[ \rho^2 \tau^2 = (\alpha^2 + \beta^2)\tau^2 < \zeta^2 < \left(\frac{\pi}{2}\right)^2, \]

that is, \( 0 < \rho \tau < \zeta < \frac{\pi}{2} \). It follows from the monotonicity of \( y \sin y \) that

\[ \rho \tau \sin (\rho \tau) < \zeta \sin \zeta = \alpha \tau = \rho \tau \cos \theta = \rho \tau \sin \left(\frac{\pi}{2} - |\theta|\right). \]

Hence we get

\[ \rho \tau < \frac{\pi}{2} - |\theta|, \]

namely, (3.8).

Suppose that condition (3.8) holds. Then we can estimate

\[ 0 < \rho \tau \sin (\rho \tau) < \rho \tau \sin \left(\frac{\pi}{2} - |\theta|\right) = \zeta \sin \zeta, \]

and therefore, using again the monotonicity of \( y \sin y \), we obtain

\[ (\alpha^2 + \beta^2)\tau^2 = \rho^2 \tau^2 < \zeta^2. \]

Thus condition (3.9) is satisfied. The proof is complete.

Figure 4 shows the stability region for system (3.1) with (3.6). It is well-known that

\[(3.10) \quad 0 < \alpha \tau < \frac{\pi}{2} \]
is a necessary and sufficient condition for the asymptotic stability of the zero solution of the scalar differential-difference equation

\[
\dot{x}(t) = -ax(t - \tau)
\]

to be asymptotically stable. This result corresponds to the segment

\[
\left\{ (\theta, \rho \tau) : \theta = 0 \quad \text{and} \quad 0 < \rho \tau < \frac{\pi}{2} \right\}
\]

in Figure 4.

Next we consider the \(n\)-dimensional system

\[
\dot{x}(t) = Ax(t - \tau)
\]

where \(A\) is an \(n \times n\) real constant matrix and \(\tau\) is a positive constant. Denote by \(-\lambda_1, -\lambda_2, \ldots, -\lambda_l\) the real eigenvalues of \(A\) and by \(-\rho_1(\cos \theta_1 \pm i \sin \theta_1), \ldots, -\rho_m(\cos \theta_m \pm i \sin \theta_m)\) the complex eigenvalues of \(A\), where \(i^2 = -1\) and \(l + 2m = n\). Then we have:

**Theorem 3.4.** The zero solution of (3.12) is asymptotically stable if and only if

\[
0 < \lambda_j \tau < \frac{\pi}{2} \quad (j = 1, \ldots, l)
\]
and

(3.14) \quad 0 < \rho_k \tau < \frac{\pi}{2} - |\theta_k| \quad (k = 1, \ldots, m).

To prove Theorem 3.4, we need Lemma 3.1 below on the asymptotic stability for the system of differential-difference equations

(3.15) \quad \begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} P & R \\ O & Q \end{pmatrix} \begin{pmatrix} u(t - \tau) \\ v(t - \tau) \end{pmatrix}

where $P$, $Q$ and $R$ are $p \times p$, $q \times q$ and $p \times q$ real constant matrices, respectively, and $O$ is $q \times p$ zero matrix. The proof is obvious and omitted.

**Lemma 3.1.** The zero solution of (3.15) is asymptotically stable if and only if the zero solutions of

\[ \dot{w}(t) = Pw(t - \tau) \]

and

\[ \dot{w}(t) = Qw(t - \tau) \]

are asymptotically stable.

**Proof of Theorem 3.4.** Using the necessary and sufficient condition (3.10) for the scalar equation (3.11), Theorems 3.2 and 3.3, and Lemma 3.1, together with the well-known result for real Jordan canonical form, we can easily show by induction on $n$ that the zero solution of (3.12) is asymptotically stable if and only if (3.13) and (3.14) hold.

To illustrate our results on the asymptotic stability of system (3.1), we consider the following cases:

(3.16) \quad \lambda_1 = \lambda_2 = \frac{1}{e}, \quad b = -\frac{1}{e} \quad \text{and} \quad (t_0, \phi, \psi) = \left(0, \frac{5}{2} \sin 10t, \frac{5}{2} \cos 10t \right);

(3.17) \quad \lambda_1 = \lambda_2 = \frac{3}{2}, \quad b = -\frac{1}{10} \quad \text{and} \quad (t_0, \phi, \psi) = \left(0, -\frac{5}{2}, \frac{5}{2} \right);

(3.18) \quad \rho = \frac{147}{320} \pi, \quad \theta = \frac{\pi}{32} \quad \text{and} \quad (t_0, \phi, \psi) = \left(0, -t - 3, \frac{5}{2} \sin 10t \right)

where $\lambda_1$, $\lambda_2$, $b$, $\rho$ and $\theta$ are given in (3.5) and (3.6), and $(t_0, \phi, \psi)$ denotes the initial condition for (3.1). Then Theorems 3.2 and 3.3 show that the zero solution of (3.1) is asymptotically stable provided $\tau = 1$. We give some portraits of trajectories of (3.1) drawn by a computer. Figures 5–7 correspond to cases (3.16)–(3.18), respectively.
Fig. 5

Fig. 6
Stability Region for Systems of DDE

Fig. 7

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(Ricevita la 8-an de marto, 1994)
(Reviziita la 13-an de septembro, 1994)
(Reviziita la 13-an de januaro, 1995)
(Reviziita la 1-an de junio, 1995)