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Linear Differential Equation Attached to Painlevé's First Equation

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1. Introduction

Let K be an ordinary differential field of characteristic 0 with a single differentiation D, containing an element x with $Dx = 1$. Let y denote a solution of the following Painlevé's first equation

$$D^2 y = 6y^2 + x \tag{1}$$

and $R = K\langle y\rangle$. Throughout this note we suppose that equation (1) has no solution satisfying any algebraic differential equation of first order over K and the field of constants of K, say C, is algebraically closed. The linear differential equation attached to equation (1) is defined by

$$z'' = 12yz\,. \tag{2}$$

Our objective is to prove the following.

Theorem. *The Picard-Vessiot group G for equation* (2) *over R strictly agrees with $SL_2(C)$.*

It is known already that G is irreducible (cf. [4]). Our method is to show that the Riccati equation deduced from (2) by $w = z'/z$

$$w' = 12y - w^2 \tag{3}$$

has no solution algebraic over R. Our assertion results from a well-known fact about Picard-Vessiot theory (cf. Kaplanski [1]).

2. Decomposition

Let us conversely assume that equation (3) has a solution w that is algebraic over R. Let the irreducible algebraic equation for w over R be $F(w) = 0$. Though the coefficients of F are in R, we may assume that those are all in the polynomial algebra $K[y, y']$, and rewrite

$$F(w, y, y') = 0\,.$$

Thus we think of F as irreducible in $K[u, y, y']$, where u is a new indeterminate. The K-algebra $K[u, y, y']$ turns out to be a differential one by defining $Du = 12y - u^2$. Differentiating the above equation for w, we then have

$$DF(w, y, y') = 0$$

Since F is irreducible, there is a polynomial $A \in K[u, y, y']$ with

$$DF = AF .$$

The polynomial A will be seen to be linear in u.

To this end let us introduce thereupon the weight function, which is defined as

$$\omega(u^i y^j y'^k) = i + 2j + 3k$$

for power products. One remark that this extends the weight function appeared in [3]. Now we have been given a weight function, as did in [3], we decompose polynomials in $K[y, y', u]$ into some isobaric polynomials. Any nonzero polynomial $F \in K[y, y', u]$ thus has the decomposition

$$F = \sum_{h=0}^{p} F_h , \qquad F_p \neq 0 , \qquad F_h \in V_h ,$$

where V_h indicates the K-linear space spanned by power products of the same weight h. The number p is called the weight of F. It is easily seen that $\omega(DF) = \omega(F) + 1$ provided $DF \neq 0$.

Now let us observe the weights of both sides of $DF = AF$. If F has degree m in u, DF has degree $m + 1$ and weight greater by 1 than F. Therefore A is linear in u, and its coefficient of u is $-m$, which is found from the definition of D. In what follows we set $A = -mu + a$, $a \in K$, hence

$$DF = (-mu + a)F \tag{4}$$

To determine the isobaric components of F, it is required to divide the differentiation D in $K[u, y, y']$ into three parts

$$D = X + Y + Z ,$$

where

$$X = (12y - u^2)\partial/\partial u + y'\partial/\partial y + 6y^2\partial/\partial y' , \qquad Z = x\partial/\partial y' ,$$

and Y indicates the differentiation obtained by applying D to coefficients. Clearly

$$XV_{h-1} \subset V_h , \qquad YV_h \subset V_h , \qquad ZV_{h+3} \subset V_h .$$

By the use of the decomposition of F we rewrite quation (4) as

(*) $$XF_h + YF_{h+1} + ZF_{h+3} = (-mu)F_h + aF_{h+1} .$$

In order that this equation makes sense for any h, we set $F_h = 0$ for $h < 0$ or $h > p$.

3. Properties of X

In this section we regard R and $R(u)$ as differential fields with the differentiation X. Call in mind that

$$Xu = 12y - u^2 , \qquad Xy = y' , \qquad Xy' = 6y^2 .$$

The field of constants of R is known to be $L = K(\gamma)$, $\gamma = y'^2 - 4y^3$, R/K is called a Poincaré field (cf. [4]). We here however include the proof of this fact in the extended form.

Proposition 1. *If $f \in R$ satisfies $Xf = a + b/y'^2$, a, $b \in L$, then it lies in L.*

Proof. Writing $f = g + y'g$, g, $h \in L(y)$, we have

$$Xf = (4y^3 + \gamma)h_y + 6y^2h + y'g_y .$$

By assumption $g_y = 0$, hence $g \in L$. For h, we have

$$(4y^3 + \gamma)^2h_y + 6y^2(4y^3 + \gamma)h = a(4y^3 + \gamma) + b .$$

Note here that polynomial $y'^2 = 4y^3 + \gamma$ is irreducible in $L[y]$. Suppose $h \neq 0$. By degree argument we know h not be a polynomial. Let k be a prime divisor of the denominator of h. It is seen that k is nothing other than y'^2, because the orders of h_y and h in k do not agree. Hence we may set $k = 4y^3 + \gamma$. Let $-r$ be the order of h in k and write $h = k^{-r}H$, $H \in L[y]$ with H and k being coprime. Then the above equality implies

$$(6 - 12r)y^2k^{-r+1}H + k^{-r+2}H_y = ak + b ,$$

and so $r = 1$. We have $kH_y - 6y^2H = ak + b$. This equation, however, has no polynomial solution, yielding a contradiction.

Let us introduce two distinguished polynomials. Polynomial $y'u - 6y^2$ is denoted by t, and polynomial $y't$ is denoted by s. These satisfy the following:

$$Xt = -ut , \qquad Yt = 0 , \qquad Zt = xu , \qquad Xs^{-1} = 1/y'^2 .$$

Making use of these polynomials, we prove

Proposition 2. *If $f \in R(u)$ satisfies $Xf = 0$ then it lies in L.*

Proof. Suppose conversely that there is an f not contained in L with $Xf = 0$. Since the field of constants of R is the same as L, such f is not algebraic over R. Hence f, s depend algebraically over R. This time, it is known that there exists a g algebraic over R with $Xg = 1/y'^2$. Taking its trace over R, if necessary, we may assume $g \in R$. This however contradicts Proposition 1.

Proposition 3. *If $f \in R(u)$ satisfies $Xf \in L$ then it lies in L.*

Proof. If $f \in R$ this results readily from Proposition 1. Assume $f \notin R$. Since f, s^{-1} depends algebraically over R this time, by a theorem of Ostrowski, there exists an element b of L such that $g = f + bs^{-1} \in R$. Hence $Xg = Xf + b/y'^2$, which implies $Xf = b = 0$ by Proposition 1.

4. Proof of Theorem

We shall attempt to determine F_h step by step using the results in the proceeding section. We first examine equation (*) for $h = p$

$$XF_p = -muF_p .$$

Using polynomial t, we can write this as $X(F_p/t^m) = 0$. Hence $F_p = ct^m$ for some $c \in L$. Thus $c = c_0\gamma^k$ and $F_p = c_0\gamma^k t^m$ for some $c_0 \in K$ and non-negative integer k. We may assume $c_0 = 1$ without loss of generality. Note that $p = 6k + 4m \geq 4$.

For $h = p - 1$ equation (*) reads

$$XF_{p-1} = -muF_{p-1} + aF_p$$

This yields

$$X(F_{p-1}/F_p) = a \in K \subset L,$$

which implies $a = 0$ and $F_{p-1} = 0$ by Proposition 3.

From equation (*) for $h = p - 2$, $p - 3$

$$XF_{p-2} = -muF_{p-2} , \qquad XF_{p-3} = -muF_{p-3} ,$$

in the same way as above, it follows that $F_{p-2} = F_{p-3} = 0$.

For $h = p - 4$ equation (*) reads

$$XF_{p-4} + ZF_p = -myF_p .$$

The logarithmic derivative of F_p with respect to Z is

$$ZF_p/F_p = kZ\gamma/\gamma + mZt/t$$
$$= 2kxy'/\gamma + mxu/t$$
$$= X(2kxy/\gamma + mx/t)\,.$$

Hence

$$X(F_{p-4}/F_p) + X(2kxy/\gamma + mx/t) = 0\,,$$

and

$$F_{p-4} = -(2kxy/\gamma + mx/t)F_p\,,$$

in view of Proposition 2. Remark that $p > 4$. In fact, $p = 4$ would implicate $F_0 \in K$, $k = 0$, $m = 1$ and $F = t - x$. This would make a contradiction.

Finally consider equation (*) for $h = p - 5$:

$$XF_{p-5} - (2ky/\gamma + m/t)F_p = -muF_{p-5}\,.$$

Then we have

$$X(F_{p-5}/F_p) = 2ky/\gamma + mys^{-1}\,.$$

Let the expansion of F_{p-5}/F_p in s be

$$\sum_{i=r}^{\infty} a_i s^i\,, \qquad a_r \neq 0\,, \qquad a_i \in R\,.$$

Then

$$\sum_{i=r}^{\infty} (Xa_i - (i-1)y'^{-2}a_{i-1})s^i = mys^{-1} + 2ky/\gamma\,.$$

If $r \leq -2$ then $Xa_r = 0$, $Xa_{r+1} - ry'^{-2}a_r = 0$ or my. These imply $a_r \in L$ and a contradiction in virtue of Proposition 1 or its proof. Hence $r = -1$, $Xa_{-1} = my$. Let y have only a pole P, whose order is known to be 2. It is also the only pole of a_{-1}, of which the order is seen to be 1. But such an element does not exist in R. This completes the proof.

References

[1] Kaplanski, I., *An Introduction to Differential Algebra*, Hermann, Paris, 1957.
[2] Kolchin, E. R., *Differential Algebra and Algebraic Group*, Academic Press, New York, 1973.
[3] Nishioka, K., A note on the transcendency of Painlevé's first transcendent, Nagoya Math. J., **109** (1988), 63–67.

[4] ———, Irreducibility of the differential equation attached to Painlevé's first equation, Tokyo J. Math., **16** (1993), 171–177.

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