Funkcialaj Ekvacioj, 37 (1994) 531–535

Hille-Wintner Type Comparison Theorems for Nonlinear Difference Equations

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In [1], a sufficient condition is derived for the existence of a positive nondecreasing solution of a class of nonlinear difference equation of the form

$$\Delta(\Delta y_{k-1})^{\sigma} + s_k y_k^{\sigma} = 0, \qquad k = 1, 2, 3, \ldots$$

where $\sigma > 0$ (this will be assumed in the sequel). In this note, we are concerned with a Hille-Wintner type comparison theorem for the existence of a positive nondecreasing solution of a slightly more general equation

(1) $$\Delta(r_{k-1}(\Delta z_{k-1})^{\sigma}) + s_k z_k^{\sigma} = 0, \qquad k = 1, 2, 3, \ldots$$

where $r_k > 0$ for $k \geq 0$. To be more precise, we shall assume that there is a positive nondecreasing solution for a "majorant" equation of the form

(2) $$\Delta(R_{k-1}(\Delta y_{k-1})^{\sigma}) + S_k y_k^{\sigma} = 0, \qquad k = 1, 2, 3, \ldots,$$

and then show that (1) has a positive nondecreasing solution also. The full conditions on $\{R_k\}_0^{\infty}$ and $\{S_k\}_1^{\infty}$ will be stated later, here we shall only assume that $\{R_k\}$ is a positive sequence bounded by a positive constant Γ. In case $\sigma = 1$, the corresponding equations reduce to linear difference equations and a Hille-Wintner type comparison theorem for positive solutions (not positive nondecreasing) of these equations has already been given by Hooker [3].

The following result (see for example [4, Theorem 41]) will be needed in the sequel.

Lemma 0. *If* $x, y \geq 0$ *and* $p > 1$, *then* $x^p - y^p \leq p x^{p-1}(x - y)$.

We begin our study by assuming that (2) has a positive nondecreasing solution $\{y_k\}_0^{\infty}$. Then letting

(3.) $$w_k = R_k(\Delta y_k)^{\sigma}/y_k^{\sigma}, \qquad k \geq 0$$

we see that $w_k \geq 0$ for $k \geq 0$ and

(4) $$\Delta w_k + \frac{w_k}{(w_k^{1/\sigma} + R_k^{1/\sigma})^{\sigma}}\{(w_k^{1/\sigma} + R_k^{1/\sigma})^{\sigma} - R_k\} + S_{k+1} = 0, \qquad k \geq 0.$$

For convenience, we shall denote

$$\frac{x}{(x^{1/\sigma}+t^{1/\sigma})^{\sigma}}\{(x^{1/\sigma}+t^{1/\sigma})-t\}$$

by $F(t, x)$. Then (4) can be rewritten as

$$\Delta w_k + F(R_k, w_k) + S_{k+1} = 0, \qquad k \geq 0. \tag{5}$$

Note that the function $F(t, x)$ is defined on the set

$$\Omega = \{(t, x) \mid 0 < t < \Gamma, x^{1/\sigma} + t^{1/\sigma} > 0\}.$$

Note further that even though $F(t, 0) = 0$, the general behavior of $F(t, x)$ depends on the value of σ and also varies on different portions of Ω. There are two different portions which are of interest, namely,

$$\Omega_1 = \{(t, x) \in \Omega \mid x \geq 0\}$$

and

$$\Omega_2 = \{(t, x) \in \Omega \mid x^{1/\sigma} + t^{1/\sigma} > 0, x \leq 0\}$$

Since

$$F_t(t, x) = \frac{-x^{1+\sigma}}{(x^{1/\sigma}+t^{1/\sigma})^{1+\sigma}},$$

it is clear that $F_t(t, x) \leq 0$ on Ω_1 and on Ω_2 when σ is a quotient of odd positive integers. Since

$$F_x(t, x) = \frac{(x^{1/\sigma}+t^{1/\sigma})^{1+\sigma}-(t^{1/\sigma})^{1+\sigma}}{(x^{1/\sigma}+t^{1/\sigma})^{1+\sigma}},$$

it is also clear that $F_x(t, x) \geq 0$ on Ω_1, Furthermore, since

$$(x^{1/\sigma}+t^{1/\sigma})^{1+\sigma}-(t^{1/\sigma})^{1+\sigma} \leq (1+\sigma)(x^{1/\sigma}+t^{1/\sigma})^{\sigma}x^{1/\sigma}$$

by Lemma 0, we see that $F_x(t, x) \leq 0$ on Ω_2 when σ is a quotient of odd positive integers.

As a consequence, if $\{(t_n, x_n)\}_0^\infty$ is a sequence of points in Ω_1 such that $x_n \geq \hat{x} > 0$, then

$$F(t_n, x_n) \geq F(\Gamma, x_n) \geq F(\Gamma, \hat{x}).$$

From this, we see that if $\{(t_n, x_n)\}_0^\infty$ is a sequence of points in Ω_1 such that $F(t_n, x_n) \to 0$, then $x_n \to 0$. A similar conclusion also holds when σ is a quotient of odd positive integers.

Lemma 1. *If $\{(t_n, x_n)\}_0^\infty$ is a sequence in Ω_1 such that $F(t_n, x_n) \to 0$, then $x_n \to 0$. If σ is a quotient of odd positive integers and $\{(t_n, x_n)\}_0^\infty$ is a sequence in Ω such that $F(t_n, x_n) \to 0$, then $x_n \to 0$.*

Lemma 2. *Suppose the sequence $\{S_k\}_1^\infty$ satisfies*

$$\sum_{k=1}^{\infty} S_k < \infty. \tag{6}$$

Then (2) *has a positive nondecreasing solution $\{y_k\}_0^\infty$ if and only if there is a nonnegative sequence $\{w_k\}_0^\infty$ which satisfies*

$$w_k = \sum_{i=k}^{\infty} F(R_i, w_i) + \sum_{i=k}^{\infty} S_{i+1} \qquad k \geq 0. \tag{7}$$

Proof. By summing (5) from $k = n$ to N, we have

$$w_{N+1} - w_n + \sum_{k=n}^{N} F(R_k, w_k) = -\sum_{k=n}^{N} S_{k+1} \qquad n \geq 0. \tag{8}$$

If

$$\sum_{k=n}^{\infty} F(R_k, w_k) = +\infty$$

then w_{N+1} diverges to $-\infty$ in view of (6). But this is impossible since $w_k^{1/\sigma} \geq 0$. Thus

$$\sum_{k=n}^{\infty} F(R_k, w_k) < +\infty$$

which implies $F(R_k, w_k)$ converges to zero. Thus by Lemma 1, $w_k \to 0$ so that taking limit on both sides of (8) as N approaches positive infinity, we obtain (7) as required.

Conversely, if $\{w_k\}_0^\infty$ is a nonnegative sequence which satisfies (7), then taking difference of both sides of (7) leads immediately to equation (4). Writing (3) in the form

$$y_{k+1}/y_k = 1 + R_k^{-1/\sigma} w_k^{1/\sigma} \qquad k \geq 0,$$

we may easily verify that the sequence $\{u_k\}_0^\infty$, defined by $u_0 = 1$ and $u_{k+1} = u_k(1 + R_k^{-1/\sigma} w_k^{1/\sigma})$ for $k \geq 0$, is a positive nondecreasing solution of (2). Q.E.D.

We are ready for the main theorem of our note.

Theorem 1. *Suppose $0 < R_k \leq r_k$ and $R_k \leq \Gamma$ for all $k \geq 0$ and some constant Γ. Suppose further that*

$$0 \le \sum_{k=n}^{\infty} s_k \le \sum_{k=n}^{\infty} S_k < \infty \tag{9}$$

for all sufficiently large n. If equation (2) *has a positive nondecreasing solution, so does equation* (1).

Proof. By Lemma 2, equation (7) has a nonnegative solution $\{w_k\}_0^\infty$. In view of the same Lemma, to prove our theorem, it suffices to find a nonnegative solution to

$$u_n = \sum_{k=n}^{\infty} F(r_k, u_k) + \sum_{k=n}^{\infty} s_{k+1}, \qquad n \ge 0. \tag{10}$$

We shall use a successive approximation argument to accomplish this. Define a sequence of successive approximation $\{u^j\}_0^\infty$ as folows:

$$u_k^0 = w_k, \qquad k \ge 0$$

$$u_k^{j+1} = \sum_{i=k}^{\infty} F(r_i, u_i) + \sum_{i=k}^{\infty} s_{i+1}, \qquad k \ge 0, j \ge 0. \tag{11}$$

Since $F(r_i, u_i^0) \le F(R_i, u_i^0)$ by Lemma 1, thus

$$0 \le u_k^1 = \sum_{i=k}^{\infty} F(r_i, u_i^0) + \sum_{i=k}^{\infty} s_{i+1} \le \sum_{i=k}^{\infty} F(R_i, u_i^0) + \sum_{i=k}^{\infty} S_{i+1} = w_k = u_k^0, \qquad k \ge 0.$$

Proceeding inductively, we assume that $0 \le u_k^j \le u_k^{j-1}$ for $k \ge 0$. Then

$$0 \le u_k^{j+1} = \sum_{i=k}^{\infty} F(r_i, u_i^j) + \sum_{i=k}^{\infty} s_{i+1} \le \sum_{i=k}^{\infty} F(R_i, u_i^{j-1}) + \sum_{i=k}^{\infty} s_{i+1} = u_k^j, \qquad k \ge 0.$$

Thus u_k^j is nonnegative and nonincreasing in j for each $k \ge 0$ so that we may define

$$u_k = \lim_j u_k^j, \qquad k \ge 0.$$

Since $0 \le u_k \le u_k^j \le u_k^0 = w_k$ for all $j \ge 1$ and $k \ge 0$, and since

$$F(r_i, u_i^j) \le F(R_i, u_i^j) \le F(R_i, u_i^0)$$

the convergence of the series in the first sum of (11) is uniform respect to j. Taking limit on both sides of (11), we then obtain (10) as required. Q.E.D.

As an example, let us quote a result in [1] which states that equation (2) has a positive nondecreasing solution when $R_k = 1$ for $k \ge 0$ and $S_k \ge 0$ for $k \ge 1$ and

$$\sum_{k=n+1}^{\infty} S_k \leq \frac{1}{e^{2\sigma} 2^{n\sigma}}, \qquad n \geq 0.$$

Combining this and Theorem 1, we see that (1) has a positive nondecreasing solution if $r_k \geq 1$ for $k \geq 0$ and

$$\sum_{k=n+1}^{\infty} s_k \leq \frac{1}{e^{2\sigma} 2^{n\sigma}}, \qquad n \geq 0.$$

Up to now we have assumed that σ is an arbitrary positive number. Suppose σ is a quotient of two odd positive integers. Then a slight variation of the proof of Lemma 2 leads readily to

Lemma 3. *Suppose σ is a quotient of two positive odd integers. Suppose further that* (6) *holds. Then* (2) *has a positive solution if and only if there exists a sequence $\{w_k\}_0^\infty$ which satisfies $w_k^{1/\sigma} + R_k^{1/\sigma} > 0$ for $k \geq 0$ and equation* (7).

By means of Lemma 3, we can then obtain a generalization of the result of Hooker mentioned earlier.

Theorem 2. *Under the same assumptions of Theorem* 1, *suppose further that σ is a quotient of two positive odd integers, then* (1) *has a positive solution when* (2) *does.*

Finally, we remark that when σ is a quotient of positive integers with odd denominator and even numerator, our previous derivation still goes through even if we do not assume $\Delta y_k \geq 0$ in (3). Consequently, Theorem 2 still holds when the value of σ is replaced by the above stated quotient.

References

[1.] Cheng, S. S. and Patula, W. T., An existence theorem for a nonlinear differential equation, to appear in Nonlinear Anal..

[2] Swanson, C., *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

[3] Hooker, J., A Hille-Wintner type comparison theorem for second order difference equations, Internat. J. Math. Math. Sci., **6** (1983), 387–394.

[4] Hardy, G. H., Littlewood J. E. and Polya, G., *Inequalities*, 2nd Ed., Cambridge University Press, 1952.

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(Ricevita la 13-an de novembro, 1992)