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Linearized Stability for Abstract Quasilinear Evolution Equations of Parabolic Type

By

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Dedicated to Professor Hiroki Tanabe on his 60th birthday

1. Introduction

We study the linearized stability of an abstract quasilinear evolution equation

$$\text{(Q)} \qquad \begin{cases} du/dt + A(u)u = f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases}$$

of parabolic type in a Banach space X. Here, $-A(u)$ are the generators of analytic semigroups on X which are defined for u in $K = \{u \in Z;\ \|u\|_Z < R\}$, $0 < R < \infty$, Z being another Banach space continuously embedded in X with $\|\cdot\|_X \leq \|\cdot\|_Z$. The domains $\mathscr{D}(A(u))$ (which may not be dense in X) are allowed to vary with u. $f(u)$ is an X-valued function defined for $u \in K$ such that $f(0) = 0$. And u_0 is an initial value in K.

The principle of linearized stability is well known as a fundamental theorem in the theory of Dynamical System of differential equation. The principle can be generalized to the abstract parabolic equations in Banach spaces. Indeed, consider an equation

$$\text{(1.1)} \qquad \begin{cases} du/dt + Au = f(u), & 0 < t < \infty, \\ u(0) = u_0 \end{cases}$$

in a Banach space X, where $-A$ is the generator of an analytic semigroup on X and f is a Hölder continuous function defined on the domain $\mathscr{D}(A^\alpha)$ of the fractional power A^α, $0 < \alpha < 1$. If A satisfies the spectral (or resolvent) condition

$$\text{(Sp)} \qquad \rho(A) \supset \{\lambda \in \boldsymbol{C};\ \operatorname{Re} \lambda \leq \delta\}, \qquad \delta > 0,$$

and if f vanishes at $u = 0$ with $\dfrac{\|f(u)\|_X}{\|A^\alpha u\|_X} \to 0$ as $u \to 0$, then it is proved that the 0 stationary solution to (1.1) is asymptotically stable (c.f. von Wahl [24], Hoshino and Yamada [8] etc.).

This result can be generalized furthermore to the quasilinear equations. Consider now an equation of the form (Q) described above. Potier-Ferry [12] has first proved the asymptotic stability under the spectral condition that the linearized operator $A(0)$ of $A(u)u$ at $u=0$ satisfies (Sp); he handled the case that $A(u)$ have a constant definition domain. His result was refined afterward by Lunardi [11]; among others, she removed a condition that $A(u)$ are densely defined operators.

When the domains of $A(u)$ vary with u, the problem becomes more difficult; we have to estimate, not only the difference of operators $A(u)$ and $A(0)$, but also the difference of the domains $\mathscr{D}(A(u))$ and $\mathscr{D}(A(0))$. The first result in this case was obtained by Drangeid [5]; according to Amann's device [1, 2, 3, 4], she reduced the problem to that of the constant domains of operators by making an assumption that suitable interpolation and extrapolation spaces of $\mathscr{D}(A(u))$ and X are independent of u. In this paper, however, we would like to present an alternative technique for handling the problem which is based on that introduced by the second author in the previous papers [16, 18, 19].

Instead of independence of the interpolation or extrapolation spaces, we shall assume simply some decay and Lipschitz condition on the resolvent $(\lambda - A(u))^{-1}$ (see (A.ii) below), which is verified in application without much effort by the theory of Function Spaces. The authors believe that in this point our technique has an advantage, because, to characterize the interpolation spaces of $\mathscr{D}(A(u))$ and X to verify the independence of u, we have to appeal not only to the theory of Function Spaces but also to the Fourier Analysis.

Before announcing our assumptions precisely, we need to introduce two more Banach spaces Y_i, $i=1, 2$, such that $Z \subset Y_i \subset X$. Then we shall make the following Assumptions.

(A.i) The resolvent sets $\rho(A(u))$ of $A(u)$, $u \in K$, contain a sector $\Sigma = \{\lambda \in \boldsymbol{C};\ |\arg(\lambda-\omega)| \geq \theta_0$ or $\lambda = \omega\}$, where $-\infty < \omega < \infty$ and $0 < \theta_0 < \pi/2$, and there the resolvents satisfy:

$$\|(\lambda - A(u))^{-1}\|_{\mathscr{L}(X)} \leq M/(|\lambda - \omega| + 1), \qquad \lambda \in \Sigma,\ u \in K,$$

with some constant M.

(A.ii) For some $0 < \nu_i \leq 1$, $i = 1, 2$,

$$\|(\omega - A(u))(\lambda - A(u))^{-1}\{(\omega - A(u))^{-1} - (\omega - A(v))^{-1}\}\|_{\mathscr{L}(X)} \leq N \frac{\|u - v\|}{(|\lambda - \omega| + 1)^{\nu_i}}, \qquad \lambda \in \Sigma, \quad u, v \in K,$$

with some constant N.
(Sp) The resolvent set $\rho(A(0))$ contains a half plane $\Lambda = \{\lambda \in \boldsymbol{C};\ \mathrm{Re}\,\lambda \leq \delta\}$ with some $\delta > 0$.
(S.i) For some $0 < \gamma_i < 1$, $\|\cdot\|_{Y_i} \leq \|\cdot\|_X^{\gamma_i}\|\cdot\|_Z^{1-\gamma_i}$, $i = 1, 2$, on Z.
(S.ii) There is some $0 < \alpha < 1$ such that the domains of the fractional powers $[A(u) - \omega]^{\alpha}$, $u \in K$, are contained in Z with continuous embedding: $\|\cdot\|_Z \leq D\,\|[A(u) - \omega]^{\alpha}\cdot\|_X$ with some constant D.
(S.iii) There are some $0 < \alpha_i < 1$ such that the domains of the fractional powers $[A(u)-\omega]^{\alpha_i}$, $u \in K$, are contained in Y_i, $i = 1, 2$, with $\|\cdot\|_{Y_i} \leq D_i\,\|[A(u)-\omega]^{\alpha_i}\cdot\|_X$ with some constants D_i.
(S.iv) The unit ball $\{u \in Z;\ \|u\|_Z \leq 1\}$ of Z is a closed subset of X.

(f.i) $$\|f(u) - f(v)\|_X \leq L\|u - v\|_{Y_2}, \qquad u, v \in K, \text{ with some constant } L.$$

(f.ii) $$\frac{\|f(u)\|_X}{\|u\|_Z} \longrightarrow 0 \quad \text{as} \quad u \longrightarrow 0 \text{ in } Z.$$

(Ex) The exponents satisfy the relations: $\alpha\gamma_1 + \nu_i > 1$, $i = 1, 2$, and $\alpha_2 < \nu_2$.
(In) The ininial value u_0 belongs to $\mathscr{D}([A(u_0)]^{\alpha})$ $(\subset Z)$.
(As will be noticed below, it can be assumed without loss of generality that $\omega = 0$.)

The coming two sections will be devoted to some preparations. In Section 2, we shall consider some related linear equations and shall obtain by virtue of (Sp) the exponential decay estimates for the fundamental solutions the way of construction of which has been established in [16, 18]. In Section 3, the local existence of solution to (Q) will be surveyed, which was obtained in [19] under (A.i, ii), (S.i, ii, iii, iv), (f.i), (Ex) and (In). The main results will be proved in Sections 4 and 5; first, the global existence of solution will be shown by utilizing (f.ii) for sufficiently small initial value u_0; then, their exponential decay will be established.

As an application we shall handle in Section 6 a quasilinear parabolic partial differential equation

$$\text{(D)} \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left\{a_{ij}(x, v)\frac{\partial v}{\partial x_j}\right\} = f(x, v, \nabla v) & \text{in } (0, \infty) \times \Omega, \\ \displaystyle\sum_{i,j=1}^{n} a_{ij}(x, v)\nu_i(x)\frac{\partial v}{\partial x_j} + g(x, v)v = 0 & \text{on } (0, \infty) \times \partial\Omega, \end{cases}$$

in a bounded region $\Omega \subset \boldsymbol{R}^n$. For the stationary solutions $\bar{u}$ to (D), it will be shown that the spectral condition (Sp) for the linearized operators at $v = \bar{u}$ is sufficient in order that they are asymptotically stable in the L^p-space $(n < p < \infty)$. To authors' knowledge there is very little literature, except Drangeid [5], on the stability of differential equations of the form (D); when

$f=g=0$, the asymptotic behavior of solution was studied by Kawanago [9, 10].

As we are working in a general setting, our abstract results are equally applicable to more general differential equations. For example, it is possible to apply them to the strongly coupled parabolic system in Population Dynamics presented by Shigesada et al. [13] the global existence for which was recently studied in [20, 21] by the similar technique of utilizing the abstract equations.

Notations. The norm of a Banach space is denoted by $\|\cdot\|_X$. $\mathscr{L}(X, Y)$ is, for two Banach spaces, the Banach space of all bounded linear operators from X to Y with the uniform operator norm $\|\cdot\|_{\mathscr{L}(X,Y)}$; $\mathscr{L}(X, X)$ will be abbreviated as $\mathscr{L}(X)$. Let $I \subset \boldsymbol{R}$ be an interval; $\mathscr{C}(I; X)$, $\mathscr{C}^\eta(I; X)$ $(0<\eta<1)$ and $\mathscr{C}^1(I; X)$ denote respectively the function spaces of continuous, of Hölder continuous with exponent η and of continuously differentiable functions defined on I with values in a Banach space X. $\mathscr{B}(I; X)$ is the function space of bounded functions on I with values in X. By C we denote a universal constant which is determined in each occurrence by the quantities occurring in the Assumptions (A.i, ii), (Sp), (S.i, ii, iii, iv), (f.i, ii) and (In) in a specific way. When the constant C depends on some parameter, say θ, it will be denoted by C_θ.

We conclude by noting:

Proposition 1.1. *The Condition* (Sp) *jointed with* (A.i, ii) *and* (S.i) *implies that, if* $0<r<R$ *is sufficiently small, then* $\rho(A(u))$ *contain, for all* $\|u\|_Z<r$, *the half plane* Λ.

Proof. When $\delta \le \omega$, there is nothing to prove; so let $\omega<\delta$. For $\lambda \in \Sigma$, the resolvent equation reads

$$\begin{aligned}(\lambda-A(u))^{-1}-(\lambda-A(0))^{-1} = (\omega-A(u))(\lambda-A(u))^{-1}\{(\omega-A(u))^{-1} \\ -(\omega-A(0))^{-1}\}(\omega-A(0))(\lambda-A(0))^{-1}.\end{aligned}$$

After some calculation,

$$\begin{aligned}&(\lambda-A(u))^{-1}\{1+(\lambda-\omega)D(u)(\omega-A(0))(\lambda-A(0))^{-1}\} \\ &\quad=(\lambda-A(0))^{-1}+D(u)(\omega-A(0))(\lambda-A(0))^{-1},\end{aligned}$$

where $D(u)=(\omega-A(u))^{-1}-(\omega-A(0))^{-1}$. So that,

$$\begin{aligned}(1.2)\qquad (\lambda-A(u))^{-1} &= \{1+(\lambda-\omega)D(u)(\omega-A(0))(\lambda-A(0))^{-1}\}^{-1} \\ &\quad\times\{(\lambda-A(0))^{-1}+D(u)(\omega-A(0))(\lambda-A(0))^{-1}\},\end{aligned}$$

provided that

(1.3) $$\|(\lambda-\omega)D(u)(\omega-A(0))(\lambda-A(0))^{-1}\|_{\mathscr{L}(X)}<1.$$

On the other hand, since (A.ii) ($\lambda=\omega$) and (Sp) yield that $\|D(u)\|_{\mathscr{L}(X)}\le N\|u\|_Z$, (1.3) can hold for all $\lambda\in\Lambda-\Sigma$, provided that the norm $\|u\|_Z<r$ is sufficiently small. This then means that for such u the right hand term of (1.2) is analytic in the triangle $\Lambda-\Sigma$; that is, $(\lambda-A(u))^{-1}$ has an analytic continuation over $\Lambda-\Sigma$.

Since we are concerned only with solutions lying in the small neighborhood of the zero solution, it is allowed to take K arbitrarily small without loss of generarity. This then means that we can assume, and in fact do assume, that

(Sp)

$\rho(A(u))\supset\Lambda$ for all $u\in K$; and $\|(\lambda-A(u))^{-1}\|_{\mathscr{L}(X)}\le M$ for $\lambda\in\Lambda$, $u\in K$.

Similarly, replacing θ_0 and some constants if necessary, we can and do assume that the Assumptions (A.i, ii) and (S.ii, iii) hold with $\omega=0$.

The authors would like to dedicate this paper to Professor Hiroki Tanabe on the occasion of his 60th birthday who has impressed on them the breadth and the depth of the study of Equations of Evolution by his introductive lectures and by his book [22] which have occasioned them to find their standing points in the researches.

2. Related linear equations

In this section we shall consider a linear evolution equation

(L) $$\begin{cases} du/dt+A(t)u=f(t), & 0<t\le T,\\ u(0)=u_0 \end{cases}$$

in X. Here, $-A(t)$, $0\le t\le T$, are the generators of analytic semigroups on X; $f\colon[0,T]\to X$ is a Hölder continuous function; and $u_0\in X$ is an initial value.

We shall assume the following Conditions.

(L.A.i) The resolvent sets $\rho(A(t))$ $(0\le t\le T)$ of $A(t)$ contain $\Sigma=\{\lambda\in\boldsymbol{C};\ |\arg\lambda|\ge\theta_0\}$, $0<\theta_0<\pi/2$, and there the resolvents satisfy:

$$\|(\lambda-A(t))^{-1}\|_{\mathscr{L}(X)}\le M/(|\lambda|+1),\ \lambda\in\Sigma,\ 0\le t\le T,$$

with some constant M.

(L.A.ii) For some $0<\mu,\nu\le 1$,

$$\|A(t)(\lambda-A(t))^{-1}\{A(t)^{-1}-A(s)^{-1}\}\|_{\mathscr{L}(X)}\le N\frac{|t-s|^{\mu}}{(|\lambda|+1)^{\nu}},$$

$$\lambda\in\Sigma,\ 0\le s,\ t\le T,$$

with some constant N.
(L.Sp) $\rho(A(t)) \supset \Lambda = \{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \leq \delta\}$, $\delta > 0$, for $0 \leq t \leq T$; and $\|(\lambda - A(t))^{-1}\|_{\mathscr{L}(X)} \leq M$ for $\lambda \in \Lambda$, $0 \leq t \leq T$.
(L.Ex) The exponents satisfy a relation: $\mu + \nu > 1$.

According to [16, 18], a fundamental solution $U(t, s)$, $0 \leq s \leq t \leq T$, called often evolution operator for $A(t)$, can be constructed under these (L.A.i, ii) and (L.Ex). And the formula

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau)f(\tau)d\tau, \qquad 0 \leq t \leq T,$$

gives a unique X-valued $\mathscr{C}^1$-solution to (L).

The goal of this section is then to establish various decay estimates for $U(t, s)$ which we need in the subsequent sections.

The following Proposition will be essential in our argument.

Proposition 2.1. *Let $\varphi(t, s)$ be a real valued continuous function defined for $0 \leq s < t \leq T$ which has a weak integrable singularity at $t = s$. And let φ satisfies an integral inequality either*

$$\varphi(t, s) \leq A(t - s)^{a-1}e^{-d(t-s)} + B\int_s^t (t - \tau)^{b-1}e^{-d(t-\tau)}\varphi(\tau, s)d\tau \tag{2.1}$$

or

$$\varphi(t, s) \leq A(t - s)^{a-1}e^{-d(t-s)} + B\int_s^t \varphi(t, \tau)(\tau - s)^{b-1}e^{-d(\tau-s)}d\tau \tag{2.2}$$

for all $0 \leq s < t \leq T$, with some constants $A, B \geq 0$ and some exponents $0 < a, b \leq 1$ and $d > 0$. Then, φ is estimated by

$$\begin{aligned}\varphi(t, s) \leq C_{a,b}A[&(t - s)^{a-1}e^{-d(t-s)} \\ &+ \{B\Gamma(b)\}^{(1-a)/b}\exp([\{B\Gamma(b)\}^{1/b} - d](t - s))], \quad 0 \leq s < t \leq T,\end{aligned} \tag{2.3}$$

with some constant $C_{a,b}$ determined by a and b alone, here $\Gamma(\cdot)$ is the gamma function.

Proof. Since the proofs are quite analogous in both cases, we may consider only the case when (2.1) is satisfied. By induction on $n = 0, 1, 2, \cdots$, it is easily verified that (2.1) implies

$$\begin{aligned}\varphi(t, s) \leq &\sum_{k=0}^{n} AB^k \frac{\Gamma(a)\Gamma(b)^k}{\Gamma(a + kb)}(t - s)^{a+kb-1}e^{-d(t-s)} \\ &+ B^{n+1}\frac{\Gamma(b)^{n+1}}{\Gamma((n + 1)b)}\int_s^t (t - \tau)^{(n+1)b-1}e^{-d(t-\tau)}\varphi(\tau, s)d\tau.\end{aligned}$$

Since $\dfrac{x^\sigma}{\Gamma(\sigma)} \to 0$ as $\sigma \to \infty$ for any $x \geq 0$ (cf. (2.5)), the integral term is seen to tend to 0 as $n \to \infty$. So that, it follows that

$$\varphi(t, s) \leq A\Gamma(a)(t-s)^{a-1}e^{-d(t-s)}E(\{B\Gamma(b)\}^{1/b}(t-s)), \tag{2.4}$$

where $E(x) = \sum_{k=0}^{\infty} \dfrac{x^{kb}}{\Gamma(a+kb)}$ is a function defined on $[0, \infty)$. While we observe:

Lemma 2.2.

$$E(x) \leq C_{a,b}(1 + x^{1-a}e^x) \qquad \text{for } 0 \leq x < \infty,$$

with some constant $C_{a,b} > 0$.

If this is observed to be true, we shall conclude from (2.4) the desired estimate (2.3).

Proof of Lemma. Let $\ell \geq 1$ denote any positive integer and σ any real number such that $\ell \leq \sigma < \ell + 1$. Note first that

$$\frac{x^{\sigma-1}}{\Gamma(\sigma)} \leq \frac{1}{\Gamma_0}\left\{\frac{x^{\ell-1}}{(\ell-1)!} + \frac{x^\ell}{\ell!}\right\}, \qquad x \geq 0, \tag{2.5}$$

with $\Gamma_0 = \operatorname*{Min}_{\sigma \geq 1} \Gamma(\sigma)$. Indeed, when $\ell = 1$, this is observed directly. If we integrate (2.5) from 0 to x, then the same inequality for $\ell + 1 \leq \sigma < \ell + 2$ is obtained; so, for $\ell \geq 2$, it suffices to use the induction. Note next that the number of k such that $\ell \leq a + kb < \ell + 1$ does not exceed $1/b$. Therefore, writing:

$$E(x) = \sum_{a+kb<1} \frac{x^{kb}}{\Gamma(a+kb)} + x^{1-a} \sum_{\ell=1}^{\infty} \sum_{\ell \leq a+kb<\ell+1} \frac{x^{a+kb-1}}{\Gamma(a+kb)},$$

we conclude the result.

Using the Proposition we establish:

Theorem 2.3. *Let* (L.A.i, ii), (L.Sp) *and* (L.Ex) *be satisfied, and fix a* ρ *such that* $1 - \mu < \rho < \nu$. *Then, for any* $0 < \delta' < \delta$, *the estimates*:

$$\|A(t)^\theta U(t, s)\|_{\mathscr{L}(X)} \leq C_{\delta'}(t-s)^{-\theta} \exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\}, \quad 0 \leq \theta \leq 1, \tag{2.6}$$

$$\|A(t)^\theta U(t, s)A(s)^{-\theta}\|_{\mathscr{L}(X)} \leq C_{\delta'} \exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\}, \quad 0 \leq \theta \leq 1, \tag{2.7}$$

(2.8) $$\|\{U(t, s) - 1\}A(s)^{-\theta}\|_{\mathscr{L}(X)} \le C_{\delta'}(t-s)^{\theta}[\exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\} + 1], \qquad 0 \le \theta \le 1,$$

(2.9) $$\|A(t)U(t, s) - A(t)\exp(-(t-s)A(t))\|_{\mathscr{L}(X)} \le C_{\delta'}(t-s)^{\mu+\nu-2}\exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\},$$

hold for $0 \le s < t \le T$ *with some* $C_{\delta'}$ *which is independent of* N *and* T.

Proof. Let us first verity (2.6) with $\theta = 0$. It is known ([18, (3.2)]) that $U(t, s)A(s)^{1-\rho}$, $1 - \mu < \rho < \nu$, is a solution to an integral equation

$$U(t, s)A(s)^{1-\rho} = A(s)^{1-\rho}\exp(-(t-s)A(s)) + \int_s^t U(t, \tau)A(\tau)^{1-\rho}A(\tau)^{\rho}\{A(\tau)^{-1} - A(s)^{-1}\}A(s)^{2-\rho}\exp(-(\tau - s)A(s))d\tau.$$

While (L.Sp) yields that, for any $0 < \delta' < \delta$,

$$\|A(s)^{\theta}\exp(-\tau A(s))\|_{\mathscr{L}(X)} \le C_{\delta'}\tau^{-\theta}e^{-\delta'\tau}, \ \tau > 0, \qquad 0 \le \theta \le 2, \ 0 \le s \le T,$$

$C_{\delta'}$ being independent of T. In addition, (L.A.ii) yields ([18, (2.2)]) that

$$\|A(t)^{\rho}\{A(t)^{-1} - A(s)^{-1}\}\|_{\mathscr{L}(X)} \le CN|t-s|^{\mu}, \qquad 0 \le s, \ t \le T,$$

C being independent of N and T. Then, by virtue of Proposition 2.1, we obtain that

$$\|U(t, s)A(s)^{1-\rho}\|_{\mathscr{L}(X)} \le C_{\delta'}(t-s)^{\rho-1}\exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\},$$

$C_{\delta'}$ being independent of N and T. Since $U(t, s) = U(t, s)A(s)^{1-\rho} \times A(s)^{\rho-1}$, this shows that (2.6), $\theta = 0$, holds when $t - s \ge 1$; on the other hand, it is nothing to speak of when $t - s \le 1$.

If $t - s$ remains bounded, say ≤ 1, all the estimates to be shown are already known by [19, Sec. 2] (indeed by [19; (2.5), (2.13), (2.11) and Theorem 2.4]). So we have only to consider the case when $t - s > 1$. But then all the estimates are obtained from the above particular estimate. For example,

$$\|A(t)^{\theta}U(t, s)\|_{\mathscr{L}(X)} \le \|A(t)^{\theta}U(t, t-1)\|_{\mathscr{L}(X)}\|U(t-1, s)\|_{\mathscr{L}(X)} \le C_{\delta'}\exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\}, \qquad t - s \ge 1.$$

It is the same of (2.7), (2.8) and (2.9).

We shall conclude this section by observing an application of Theorem 2.3 to our original quasilinear problem.

Consider a function u: $[0, T] \to Z$ satisfying:

(2.10) $$\begin{cases} \|u(t)\|_Z \le r, & 0 \le t \le T, \\ \|u(t) - u(s)\|_X \le k|t-s|^{\eta}, & 0 \le s, \ t \le T, \end{cases}$$

with some $0 < r < R$, $k > 0$ and $(1 - \nu_1)/\gamma_1 < \eta < \alpha$. From such a u, linear operators $A_u(t) = A(u(t))$, $0 \le t \le T$, are defined. Then, (A.i) implies that $A_u(t)$ satisfy the Condition (L.A.i). Similarly, (A.ii) (i = 1) together with (S.i) (i = 1) implies that

$$\| A_u(t)(\lambda - A_u(t))^{-1}\{A_u(t)^{-1} - A_u(s)^{-1}\} \|_{\mathscr{L}(X)} \le Nk^{\gamma_1}(2r)^{1-\gamma_1} \frac{|t-s|^{\gamma_1 \eta}}{(|\lambda| + 1)^{\nu_1}},$$

which shows that (L.A.ii) is valid with $\mu = \gamma_1 \eta$ and $\nu = \nu_1$. Since (L.Ex) is trivial, there exists an evolution operator $U_u(t, s)$ for $A_u(t)$. Moreover, since (Sp) implies (L.Sp), we observe:

Corollary 2.4. *Under* (A.i, ii), (Sp), (S.i) *and* (Ex), *let u be a function satisfying* (2.10). *Let δ' be any number such that $0 < \delta' < \delta$. If r is sufficiently small, then the estimates*:

$$\| [A_u(t)]^\theta U_u(t, s) \|_{\mathscr{L}(X)} \le C_{\delta'}(t - s)^{-\theta} e^{-\delta'(t-s)}, \qquad 0 \le \theta \le 1, \tag{2.11}$$

$$\| [A_u(t)]^\theta U_u(t, s)[A_u(s)]^{-\theta} \|_{\mathscr{L}(X)} \le C_{\delta'} e^{-\delta'(t-s)}, \qquad 0 \le \theta \le 1, \tag{2.12}$$

$$\| \{U_u(t, s) - 1\} [A_u(s)]^{-\theta} \|_{\mathscr{L}(X)} \le C_{\delta'}(t - s)^\theta, \qquad 0 \le \theta \le 1, \tag{2.13}$$

$$\begin{aligned} &\| A_u(t) U_u(t, s) - A_u(t) \exp(-(t - s) A_u(t)) \|_{\mathscr{L}(X)} \\ &\quad \le C_{\delta'}(t - s)^{\gamma_1 \eta + \nu_1 - 2} e^{-\delta'(t-s)}, \end{aligned} \tag{2.14}$$

hold for $0 \le s < t \le T$ with some $C_{\delta'}$ independent of T.

3. Local existence of solutions to (Q)

In the paper [19] we studied the local existence of solution to (Q) under the Assumptions (A.i, ii), (S.i, ii, iii, iv), (f.i), (Ex) and (In). In this section we shall survey the results with adding necessary modifications for the present problem.

Let us first verify:

Proposition 3.1. *Under* (A.i, ii), (S.i, ii, iii, iv), (f.i) *and* (Ex), *let $r', k' > 0$ be any positive numbers and let η' be any number such that $(1 - \nu_i)/\gamma_1 < \eta' < \alpha$ $(i = 1, 2)$. Then, there exist $\varepsilon', S > 0$ such that, for any initial value u_0 satisfying* (In), *if* $\| [A(u_0)]^\alpha u_0 \|_X \le \varepsilon'$, *then* (Q) *possesses a unique local solution $u \in \mathscr{C}^1((0, S]; X)$ on the interval $[0, S]$ in the function space*:

$$\begin{cases} \| [A(u(t))]^\alpha u(t) \|_X \le r', & 0 \le t \le S, \\ \| u(t) - u(s) \|_X \le k' |t - s|^{\eta'}, & 0 \le s, t \le S. \end{cases} \tag{3.1}$$

Proof. The proof will consist of five Steps.

Step 1. Take an r such that $0 < r < R$, and let $\|u_0\|_Z < r$. For each $0 < S < \infty$, we set:

$$\mathscr{X}(S) = \mathscr{C}(I_S; Y_2) \qquad \text{and}$$

$$\mathscr{Y}(S) = \{u \in \mathscr{C}^{\eta}(I_S; X) \cap \mathscr{B}(I_S; Z); u(0) = u_0,$$
$$\|u(t) - u(s)\|_X \le k'|t - s|^{\eta'} \text{ for } s, t \in I_S, \ \|u(t)\|_Z \le r \text{ for } t \in I_S\},$$

where $I_S = [0, S]$. Then it is easily seen that $\mathscr{Y}(S)$ is a closed subset of $\mathscr{X}(S)$; indeed, note from (S.iv) that if $\|u_n\|_Z \le r$ and $u_n \to u$ in Y_2 (and hence in X) as $n \to \infty$, then $\|u\|_Z \le r$.

Step 2. Consider a family of linear operators $A_u(t) = A(u(t))$, $t \in I_S$, defined from $u \in \mathscr{Y}(S)$. Since $u \in \mathscr{C}^{\gamma_1 \eta'}(I_S; Y_1)$ by (S.i), $A_u(t)$ are shown to satisfy the Conditions (L.A.i, ii) and (L.Ex) in Section 2 with $v = v_1$ and $\mu = \gamma_1 \eta'$; so that, there exists an evolution operator $U_u(t, s)$, $0 \le s \le t \le S$, for $A_u(t)$. By using this, a mapping Φ: $\mathscr{Y}(S) \to \mathscr{C}(I_S; X)$ is defined by

$$(\Phi u)(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, \tau) f(u(\tau)) d\tau, \qquad 0 \le t \le S.$$

Step 3. By the same argument as in the proof of [19, Theorem 3.1], it is verified that

$$\text{(3.2)} \quad \|(\Phi u)(t)\|_Z \le C\{\|[A(u_0)]^\alpha u_0\|_X + S^{1-\alpha}\}, \qquad 0 \le t \le S,$$
$$\|(\Phi u)(t) - (\Phi u)(s)\|_X \le C(t - s)^\alpha \{1 + \|[A(u_0)]^\alpha u_0\|_X\}, \qquad 0 \le s \le t \le S.$$

Therefore, if $\|[A(u_0)]^\alpha u_0\|_X$ and S are sufficiently small, then Φ maps $\mathscr{Y}(S)$ into itself.

Step 4. Similarly, by the same argument as in the proof of [19, Theorem 4.1], it is verified that, for $u, v \in \mathscr{Y}(S)$,

$$\|\Phi u - \Phi v\|_{\mathscr{X}(S)} \le C\{S^{\alpha + \rho - \alpha_2 - 1} \|[A(u_0)]^\alpha u_0\|_X + S^{\rho - \alpha_2}\} \|u - v\|_{\mathscr{X}(S)}$$

with some ρ such that $\operatorname{Max}\{\alpha_2, 1 - \gamma_1 \eta'\} < \rho < v_2$. This shows that, if $\|[A(u_0)]^\alpha u_0\|_X$ and S are sufficiently small, then Φ is a contraction mapping of $\mathscr{X}(S)$.

Step 5. In this way we observe that Φ has a unique fixed point $u \in \mathscr{Y}(S)$, which gives a solution to (Q) on the interval I_S. Finally, it is verified as for (3.2) that

$$\|[A(u(t))]^\alpha u(t)\|_X \le C\{\|[A(u_0)]^\alpha u_0\|_X + S^{1-\alpha}\}, \qquad 0 \le t \le S,$$

therefore (3.1) can be true provided that $\|[A(u_0)]^\alpha u_0\|_X$ and S are sufficiently small.

Let us now consider an intial value problem

$$(\mathrm{Q}_\sigma) \qquad \begin{cases} du/dt + A(u)u = f(u), & \sigma < t < \infty, \\ u(\sigma) = u_\sigma, \end{cases}$$

with a general initial point $\sigma \geq 0$ and an initial value $u_\sigma \in K$. As the equation is autonomus, the same assertion as the Proposition 3.1 is then true independently of σ. That is, we have proved:

Theorem 3.2. *Under* (A.i, ii), (S.i, ii, iii, iv), (f.i) *and* (Ex), *let* r', k' *and* η' *be as in Proposition* 3.1. *Then, there exist* ε', $S > 0$ *such that, for any initial value* u_σ *satisfing* (In), *if* $\|[A(u_\sigma)]^\alpha u_\sigma\|_X \leq \varepsilon'$, *then* (Q_σ) *possesses a unique local solution* $u \in \mathscr{C}^1((\sigma, \sigma + S]; X)$ *on the interval* $[\sigma, \sigma + S]$ *in the function space*:

$$(3.3) \qquad \begin{cases} \|[A(u(t))]^\alpha u(t)\|_X \leq r', & \sigma \leq t \leq \sigma + S, \\ \|u(t) - u(s)\|_X \leq k'|t - s|^{\eta'}, & \sigma \leq s,\, t \leq \sigma + S, \end{cases}$$

in particular, ε' *and* S *being independent of the initial point* σ.

4. Global existence of solutions to (Q)

On the basis of the results in the previous two Sections we shall here establish the global existence of solution to (Q) for sufficiently small initial values u_0.

We begin with verifying an a priori estimate for (Q):

Proposition 4.1. *Under* (A.i, ii), (Sp), (Si, ii, iii, iv), (f.i, ii) *and* (Ex), *one can choose numbers* R'', $K'' > 0$ *and* η'' *such that* $(1 - \nu_i)/\gamma_1 < \eta'' < \alpha$ $(i = 1, 2)$ *as the following statement holds. For any* r'', $k'' > 0$, *there exists* $\varepsilon'' > 0$ *such that, if the initial value* u_0 *satisfies* (In) *and* $\|[A(u_0)]^\alpha u_0\|_X \leq \varepsilon''$, *then every local solution* $u \in \mathscr{C}^1((0, T]; X)$ *to* (Q) *lying in the function space*:

$$(4.1) \qquad \begin{cases} \|[A(u(t))]^\alpha u(t)\|_X \leq R'', & 0 \leq t \leq T, \\ \|u(t) - u(s)\|_X \leq K''|t - s|^{\eta''}, & 0 \leq s,\, t \leq T, \end{cases}$$

must actually satisfy:

$$(4.2) \qquad \begin{cases} \|[A(u(t))]^\alpha u(t)\|_X \leq r'', & 0 \leq t \leq T, \\ \|u(t) - u(s)\|_X \leq k''|t - s|^{\eta''}, & 0 \leq s,\, t \leq T, \end{cases}$$

ε'' *being independent of* T.

Proof. Choose a $K'' > 0$ and an η'' such that $(1 - \nu_i)/\gamma_1 < \eta'' < \alpha$ $(i = 1, 2)$ arbitrarily. Then, for any solution u to (Q) satisfying (4.1), the evolution operator $U_u(t, s)$ for $A_u(t) = A(u(t))$ exists, because (4.1) implies that (L.A.i, ii) and (L.Ex) in Section 2 are fulfilled by $A_u(t)$ with $\mu = \gamma_1\eta''$ and $\nu = \nu_1$. In addition, if $R'' > 0$ is chosen sufficiently small, then by virtue of

Corollary 2.4 $U_u(t, s)$ satisfies (2.11) and (2.12) with some $\delta' > 0$. Therefore, since

$$u(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, \tau) f(u(\tau))d\tau, \qquad 0 \leq t \leq T,$$

it follows that

$$\begin{aligned}
(4.3) \qquad & \| [A_u(t)]^\alpha u(t) \|_X \\
& \leq C_{\delta'} \left\{ e^{-\delta' t} \| [A(u_0)]^\alpha u_0 \|_X + \int_0^t (t-\tau)^{-\alpha} e^{-\delta'(t-\tau)} \| f(u(\tau)) \|_X d\tau \right\} \\
& \leq C_{\delta'} \left\{ e^{-\delta' t} \| [A(u_0)]^\alpha u_0 \|_X + D_f(DR'') \int_0^t (t-\tau)^{-\alpha} e^{-\delta'(t-\tau)} \| [A_u(\tau)]^\alpha u(\tau) \|_X d\tau \right\} \\
& \leq C_{\delta'} \{ \| [A(u_0)]^\alpha u_0 \|_X + D_f(DR'') \sup_{0 \leq \tau \leq t} \| [A(u(\tau))]^\alpha u(\tau) \|_X \},
\end{aligned}$$

here we used a notation $D_f(r) = \sup_{\|u\|_Z \leq r} \{ \| f(u) \|_X / \| u \|_Z \}$, $C_{\delta'}$ being independent of T. As $D_f(r) \to 0$ as $r \to 0$ from (f.ii), this yields that, if $R'' > 0$ is sufficiently small,

$$(4.4) \qquad \sup_{0 \leq t \leq T} \| [A_u(t)]^\alpha u(t) \|_X \leq C_{\delta'} \| [A(u_0)]^\alpha u_0 \|_X,$$

which then shows the first result of (4.2). On the other hand, since

$$u(t) - u(s) = \{ U_u(t, s) - 1 \} u(s) + \int_s^t U_u(t, \tau) f(u(\tau)) d\tau, \qquad 0 \leq s, t \leq T,$$

it follows from (2.11) and (2.13) that

$$\begin{aligned}
\| u(t) - u(s) \|_X \leq C_{\delta'} \Bigg[& (t-s)^{\eta''} \| [A_u(s)]^{\eta'' - \alpha} \|_{\mathscr{L}(X)} \| [A_u(s)]^\alpha u(s) \|_X \\
& + \left\{ \int_s^t e^{-\delta'(t-\tau)} d\tau \right\} \sup_{s \leq \tau \leq t} \| [A_u(\tau)]^\alpha u(\tau) \|_X \Bigg].
\end{aligned}$$

So that, in view of (4.4),

$$\| u(t) - u(s) \|_X \leq C_{\delta'} (t-s)^{\eta''} \| [A(u_0)]^\alpha u_0 \|_X, \qquad 0 \leq s, \ t \leq T,$$

which then shows the second result of (4.2).

We can now prove:

Theorem 4.2. *Assume* (A.i, ii), (Sp), (S.i, ii, iii, iv), (f.i, ii) *and* (Ex). *Then, there exists* $\varepsilon > 0$ *such that, if the initial value* u_0 *satisfies* (In) *and*

$\|[A(u_0)]^\alpha u_0\|_X \leq \varepsilon$, *then* (Q) *possesses a unique global solution* $u \in \mathscr{C}^1((0, \infty); X)$ *on* $[0, \infty)$ *in the function space*:

$$(4.5) \qquad \begin{cases} \|[A(u(t))]^\alpha u(t)\|_X \leq r, & 0 \leq t < \infty, \\ \|u(t) - u(s)\|_X \leq k|t-s|^\eta, & 0 \leq s, t < \infty, \end{cases}$$

with some $r, k > 0$ *and* η *such that* $(1-\nu_i)/\gamma_1 < \gamma_1 < \eta < \alpha$ $(i = 1, 2)$.

Proof. Let R'', K'' and η'' be the numbers determined in the Proposition 4.1. Set r', k' and η' in the Theorem 3.2 as $r' = R''$, $k' = k''/2$ and $\eta' = \eta''$; and, in addition, $r'' = \varepsilon'$ and $k'' = K''/2$. Then the theorem is proved with $\varepsilon = \mathrm{Min}\{\varepsilon', \varepsilon''\}$, $r = \mathrm{Min}\{r'', R''\}$, $k = K''/2$ and $\eta = \eta'$. In fact, let u_0 satisfy $\|[A(u_0)]^\alpha u_0\|_X \leq \varepsilon$. By Proposition 3.1 there exists a solution $u \in \mathscr{C}^1((0, S]; X)$ on $[0, S]$ with $\|[A(u(t))]^\alpha u(t)\|_X \leq r' = R''$ and $\|u(t) - u(s)\|_X \leq k'|t-s|^{\eta'} \leq K''|t-s|^{\eta''}$; but, since $\|[A(u_0)]^\alpha u_0\|_X \leq \varepsilon''$, Proposition 4.1 then implies that u satisfies actually (4.5) on $[0, S]$. Assume next that there exists a solution $u \in \mathscr{C}^1((0, T]; X)$ satisfying (4.5) on an interval $[0, T]$. We then use Theorem 3.2 with $\sigma = T - S/2$ and $u_\sigma = u(\sigma)$; since $\|[A(u_\sigma)]^\alpha u_\sigma\|_X \leq \varepsilon'$, there exists a solution $u_\sigma \in \mathscr{C}^1((\sigma, \sigma+S]; X)$ with (3.3) on $[\sigma, \sigma+S]$. But, by the uniqueness of solution to (Q_σ), $u(t) = u_\sigma(t)$ on $[\sigma, T]$; therefore, u has been extended to a solution $\tilde{u} \in \mathscr{C}^1((0, \sigma+S]; X)$ on $[0, \sigma+S]$. It is immediate to see that this $\tilde{u}$ really satisfies (4.1) on $[0, \sigma+S]$; then, Proposition 4.1 again implies that $\tilde{u}$ satisfies actually (4.2) and hence (4.5). Thus we have verified that the solution on $[0, T]$ can be always extended to a solution on $[0, T+S/2]$ in the function space (4.5). Repeating this procedure, we shall obtain the desired result, since S was independent of T.

5. Asymptotic stability of zero solution

Under the Assumptions announced Introduction the asymptotic exponential stability is true for the zero solution to (Q). In fact, we prove:

Theorem 5.1. *Assume* (A.i, ii), (Sp), (S.i, ii, iii, iv), (f.i, ii) *and* (Ex). *Then, there exists* $\varepsilon > 0$ *such that, if the initial value* u_0 *satisfies* (In) *and* $\|[A(u_0)]^\alpha u_0\|_X \leq \varepsilon$, *then the global solution* u *to* (Q) *in the space* (4.5) *in Theorem* 4.2 *decays as, for any* $0 < \beta < \delta$,

$$(5.1) \qquad \|u(t)\|_Z + \|A(u(t))u(t)\|_X \leq C_\beta e^{-\beta t}\|[A(u_0)]^\alpha u_0\|_X, \qquad 1 \leq t < \infty,$$

C_β *being independent of* u_0.

Proof. The proof will be accomplished by three Steps.

Step 1. Let us first prove that $\|u(t)\|_Z \to 0$ as $t \to \infty$. As was verified above, the solution u lies in (4.1) in Propositin 4.1. So that, by (4.3) we have:

$$\|[A_u(t)]^\alpha u(t)\|_X \leq C_{\delta'}\left[e^{-\delta' t}\|[A(u_0)]^\alpha u_0\|_X\right.$$
$$\left.+ D_f(\sup_{t\geq 0}\|u(t)\|_Z)\int_0^t (t-\tau)^{-\alpha}e^{-\delta'(t-\tau)}\|[A_u(\tau)]^\alpha u(\tau)\|_X d\tau\right]$$

with some $\delta' > 0$, where $A_u(t) = A(u(t))$ and $D_f(r) = \sup_{\|u\|_Z \leq r}\{\|f(u)\|_X/\|u\|_Z\}$. By the Proposition 2.1 it then follows that

$$\|[A_u(t)]^\alpha u(t)\|_X \leq C_{\delta'}\exp[\{C_{\delta'}D_f(\sup_{t\geq 0}\|u(t)\|_Z)^{1/(1-\alpha)} - \delta'\}t]\|[A(u_0)]^\alpha u_0\|_X.$$

On the other hand, from Proposition 4.1 it is verified that $\sup_{t\geq 0}\|u(t)\|_Z \leq Dr \to 0$ provided $\varepsilon \to 0$. In other words, there exists some $\varepsilon > 0$ for which $\|[A(u_0)]^\alpha u_0\|_X \leq \varepsilon$ imlpies that

$$\|[A(u(t))]^\alpha u(t)\|_X \leq C_{\delta'}e^{-(\delta'/2)t}\|[A(u_0)]^\alpha u_0\|_X, \qquad 0 \leq t < \infty. \tag{5.2}$$

Step 2. Let $\beta < \delta'' < \delta$. Then, from Corollary 2.4, if $T_{\delta''}$ is sufficiently large, then the evolution operator $U_u(t, s)$, $T_{\delta''} \leq s \leq t < \infty$, for $A_u(t)$ satisfies (2.11) and (2.12) with $\delta' = \delta''$. Therefore, repeating the same argument as above on an interval $[T, \infty)$ with $T_{\delta''} \leq T < \infty$, we similarly obtain that

$$\|[A_u(t)]^\alpha u(t)\|_X$$
$$\leq C_{\delta''}\exp[\{C_{\delta''}D_f(\sup_{t\geq T}\|u(t)\|_Z)^{1/(1-\alpha)} - \delta''\}t]\|[A_u(T)]^\alpha u(T)\|_X, \qquad T \leq t < \infty.$$

This shows similarly that, if T is sufficiently large, then

$$\|[A_u(t)]^\alpha u(t)\|_X \leq C_{\delta''}e^{-\beta t}\|[A_u(T)]^\alpha u(T)\|_X, \qquad T \leq t < \infty.$$

This jointed with (5.2) then yields the first estimate of (5.1).

Step 3. Let again $\beta < \delta'' < \delta$. We have now proved that

$$\|[A_u(t)]^\alpha u(t)\|_X \leq C_{\delta''}e^{-\delta'' t}\|[A(u_0)]^\alpha u_0\|_X, \qquad 0 \leq t < \infty. \tag{5.3}$$

Let $T_{\delta''}$ be as above. Then we verify:

Lemma 5.2. *The solution u satisfies*:

$$\|u(t) - u(s)\|_X \leq C_{\delta''}(t-s)^\alpha e^{-\delta'' s}\|[A(u_0)]^\alpha u_0\|_X, \qquad T_{\delta''} \leq s \leq t < \infty.$$

Proof of lemma. Since

$$u(t) - u(s) = \{U_u(t, s) - 1\}[A_u(s)]^{-\alpha}[A_u(s)]^\alpha u(s) + \int_s^t U_u(t, \tau)f(u(\tau))d\tau,$$

it follows from (2.11), (2.13) and (5.3) that

$$\begin{aligned}\| u(t) - u(s) \|_X &\le C_{\delta''} \Big\{ (t-s)^\alpha \| [A_u(s)]^\alpha u(s) \|_X \\ &\quad + \int_s^t e^{-\delta''(t-\tau)} \| [A_u(\tau)]^\alpha u(\tau) \|_X d\tau \Big\} \\ &\le C_{\delta''} \{ (t-s)^\alpha e^{-\delta'' s} + (t-s) e^{-\delta'' t} \} \| [A(u_0)]^\alpha u_0 \|_X.\end{aligned}$$

For $t > T_{\delta''}$, write $A_u(t)u(t)$ in the form

$$\begin{aligned}A_u(t)u(t) &= A_u(t)U_u(t, T_{\delta''})u(T_{\delta''}) \\ &\quad + \int_{T_{\delta''}}^t A_u(t)U_u(t,\tau)\{f(u(\tau)) - f(u(t))\} d\tau \\ &\quad + \int_{T_{\delta''}}^t \{A_u(t)U_u(t,\tau) - A_u(t)\exp(-(t-\tau)A_u(t))\} d\tau f(u(t)) \\ &\quad + \{1 - \exp(-(t - T_{\delta''})A_u(t))\} f(u(t)) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

Then, from (2.11) and (5.3),

$$\| I \|_X \le C_{\delta''}(t - T_{\delta'''})^{-1} e^{-\delta'' t} \| [A(u_0)]^\alpha u_0 \|_X, \qquad T_{\delta''} < t < \infty.$$

By (f.i) and (S.i) (i = 2), we have:

$$\| \mathrm{II} \|_X \le C_{\delta''} \int_{T_{\delta''}}^t (t-\tau)^{-1} e^{-\delta''(t-\tau)} \| u(\tau) - u(t) \|_X^{\gamma_2} \| u(\tau) - u(t) \|_Z^{1-\gamma_2} d\tau,$$

the above lemma together with (5.3) yields here that

$$\begin{aligned}&\le C_{\delta''} e^{-\delta'' t} \left\{ \int_{T_{\delta''}}^t (t-\tau)^{\alpha\gamma_2 - 1} d\tau \right\} \| [A(u_0)]^\alpha u_0 \|_X \\ &\le C_{\delta''}(t - T_{\delta''})^{\alpha\gamma_2} e^{-\delta'' t} \| [A(u_0)]^\alpha u_0 \|_X, \qquad T_{\delta''} < t < \infty.\end{aligned}$$

On the other hand, (2.14) yields that

$$\begin{aligned}\| \mathrm{III} \|_X &\le C_{\delta''} \int_{T_{\delta''}}^t (t-\tau)^{\gamma_1 \eta + \nu_1 - 2} e^{-\delta''(t-\tau)} d\tau \| u(t) \|_Z \\ &\le C_{\delta''} e^{-\delta'' t} \| [A(u_0)]^\alpha u_0 \|_X, \qquad T_{\delta''} < t < \infty.\end{aligned}$$

Finally, it is clear that

$$\| \mathrm{IV} \|_X \le C \| u(t) \|_Z \le C_{\delta''} e^{-\delta''} \| [A(u_0)]^\alpha u_0 \|_X, \qquad T_{\delta''} < t < \infty.$$

Thus we have verified that the second estimate of (5.1) for large t, say, for $T_{\delta''} + 1 \le t < \infty$. For $1 \le t \le T_{\delta'''} + 1$, however, the proof is more immediate.

Indeed, by a similar argument we easily verify that

$$\| A_u(t)u(t) \|_X \leq C_{\delta''} \| [A(u_0)]^\alpha u_0 \|_X, \qquad 1 \leq t \leq T_{\delta'''} + 1.$$

In this way we complete the proof.

6. Application

As an application model of the abstract result obtained in the previous sections, let us consider the following quasilinear parabolic differential equation:

$$\text{(D)} \qquad \begin{cases} \partial v/\partial t + A(x, v; D)v = f(x, v, \nabla v) & \text{in } (0, \infty) \times \Omega, \\ B(x, v; D)v = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ v(0, x) = v_0(x) & \text{in } \Omega, \end{cases}$$

in a bounded region $\Omega \subset \boldsymbol{R}^n$. Here,

$$A(x, v; D)w = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left\{ a_{ij}(x, v) \frac{\partial w}{\partial x_j} \right\}$$

is a differential operator in Ω with real valued functions a_{ij} on $\bar{\Omega} \times \boldsymbol{C}$.

$$B(x, v; D)w = \sum_{i,j=1}^{n} a_{ij}(x, v) \nu_i(x) \frac{\partial w}{\partial x_j} + g(x, v)w$$

is a boundary differential operator on $\partial\Omega$ with a real valued function g on $\partial\Omega \times \boldsymbol{C}$, $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ being the outer normal vector at $x \in \partial\Omega$. $f(x, v, q)$ is a complex valued function on $\bar{\Omega} \times \boldsymbol{C} \times \boldsymbol{C}^n$. v_0 is an initial function in Ω. And $v = v(t, x)$, $(t, x) \in (0, \infty) \times \Omega$, is an unknown function.

We shall assume the following Conditions:

(Ω) Ω is a bounded region in $\boldsymbol{R}^n$ of $\mathscr{C}^2$-class;

(a.1) $a_{ij} \in \mathscr{C}^3(\bar{\Omega} \times (\boldsymbol{R} + i\boldsymbol{R}))$, $1 \leq i, j \leq n$;

(a.2) $a_{ij} = a_{ji}$ $(1 \leq i, j \leq n)$, and there exists some $\varepsilon > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x, u) q_i q_j \geq \varepsilon |q|^2, \qquad q \in \boldsymbol{R}^n,$$

for each $(x, u) \in \bar{\Omega} \times \boldsymbol{C}$;

(f) $f \in \mathscr{C}^2(\bar{\Omega} \times (\boldsymbol{R} + i\boldsymbol{R}) \times (\boldsymbol{R} + i\boldsymbol{R})^n)$; and

(g) $g \in \mathscr{C}^2(\partial\Omega \times (\boldsymbol{R} + i\boldsymbol{R}))$.

We are concerned with seeking a sufficient condition for the stationary solutions to (D) in order that they become asymptotically stable in the space $L^p(\Omega)$, $n < p < \infty$. The initial functions v_0 will be taken in $W_p^a(\Omega)$, $1 + n/p < a$.

Notations. $W_p^\sigma(\Omega)$ (resp. $W_p^\sigma(\partial\Omega)$), $\sigma \geq 0$, denotes the Sobolev space in Ω (resp. on $\partial\Omega$); in particular, $W_p^0(\Omega) = L^p(\Omega)$ (resp. $W_p^0(\partial\Omega) = L^p(\partial\Omega)$). $\mathscr{C}^m(\bar{\Omega})$ (resp. $\mathscr{C}^m(\partial\Omega)$), $m = 0, 1, 2, \cdots$, denotes the space of m-times continuously differentiable functions on $\bar{\Omega}$ (resp. on $\partial\Omega$). $\mathscr{C}^m(\bar{\Omega} \times (\boldsymbol{R} + i\boldsymbol{R})^k)$ (resp. $\mathscr{C}^m(\partial\Omega \times (\boldsymbol{R} + i\boldsymbol{R})^k)$, $m, k = 0, 1, 2, \cdots$, denotes the space of functions defined on $\bar{\Omega} \times \boldsymbol{C}^k$ (resp. $\partial\Omega \times \boldsymbol{C}^k$) which are m-times continuously differentiable in the sence of real variables. By C we shall denote a universal constant which is determined in each occurrence by the quantities occurring in (Ω), (a.1, 2), (f) and (g).

Let $\bar{u} = \bar{u}(x)$ be a stationary solution

$$\begin{cases} A(x, \bar{u}; D)\bar{u} = f(x, \bar{u}, \nabla\bar{u}) & \text{in } \Omega, \\ B(x, \bar{u}; D)\bar{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

to (D) such that $\bar{u} \in \mathscr{C}^2(\bar{\Omega})$. We rewrite the equation (D) around this solution $\bar{u}$ by changing the unknown function from v to $u = v - \bar{u}$. Then,

$$\begin{aligned} A(x, u + \bar{u}; D)(u + \bar{u}) - A(x, \bar{u}; D)\bar{u} \\ &= A(x, u + \bar{u}; D)u + A_u(x, \bar{u}; D)(u, \bar{u}) \\ &\quad + \int_0^1 \{A_u(x, \theta u + \bar{u}; D) - A_u(x, \bar{u}; D)\} d\theta (u, \bar{u}), \end{aligned}$$

where

$$A_u(x, u; D)(w_1, w_2) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left\{\frac{\partial a_{ij}}{\partial u}(x, u) w_1 \cdot \frac{\partial w_2}{\partial x_j}\right\}.$$

Similarly,

$$\begin{aligned} B(x, u + \bar{u}; D)(u + \bar{u}) - B(x, \bar{u}; D)\bar{u} \\ &= B(x, u + \bar{u}; D)u + \int_0^1 B_u(x, \theta u + \bar{u}; D) d\theta (u, \bar{u}), \end{aligned}$$

where

$$B_u(x, u; D)(w_1, w_2) = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial u}(x, u) v_i(x) w_1 \cdot \frac{\partial w_2}{\partial x_j} + \frac{\partial g}{\partial u}(x, u) w_1 w_2.$$

On the other hand,

$$\begin{aligned} &f(x, u + \bar{u}, \nabla(u + \bar{u})) - f(x, \bar{u}, \nabla\bar{u}) \\ &= \frac{\partial f}{\partial u}(x, \bar{u}, \nabla\bar{u})u + \nabla_q f(x, \bar{u}, \nabla\bar{u}) \cdot \nabla u + \int_0^1 \left\{\frac{\partial f}{\partial u}(x, \theta u + \bar{u}, \nabla(\theta u + \bar{u}))\right. \\ &\quad \left. - \frac{\partial f}{\partial u}(x, \bar{u}, \nabla\bar{u})\right\} d\theta u + \int_0^1 \nabla_q f(x, \theta u + \bar{u}, \nabla(\theta u + \bar{u})) - \nabla_q f(x, \bar{u}, \nabla\bar{u})\} d\theta \cdot \nabla u. \end{aligned}$$

Therefore, (D) is rewritten as:

$$(6.1)\qquad \begin{cases} \partial u/\partial t + \mathscr{A}(x, u; D)u = \not{f}(x, u, \nabla u) & \text{in } (0, \infty) \times \Omega, \\ \mathscr{B}(x, u; D)u = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Here,

$$\begin{aligned} \mathscr{A}(x, u; D)w &= A(x, u + \bar{u}; D)w + A_u(x, \bar{u}; D)(w, \bar{u}) \\ &\quad - \frac{\partial f}{\partial u}(x, \bar{u}, \nabla\bar{u})w - \nabla_q f(x, \bar{u}, \nabla\bar{u}) \cdot \nabla w \\ &= - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left\{a_{ij}(x, \bar{u} + u)\frac{\partial w}{\partial x_j}\right\} + \sum_{i=1}^{n} b_i(x)\frac{\partial w}{\partial x_i} + c(x)w \end{aligned}$$

is a diferential operator in Ω with functions

$$b_i(x) = - \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial u}(x, \bar{u})\frac{\partial \bar{u}}{\partial x_j} - \frac{\partial f}{\partial q_i}(x, \bar{u}, \nabla\bar{u}),$$

$$c(x) = - \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left\{\frac{\partial a_{ij}}{\partial u}(x, \bar{u})\frac{\partial \bar{u}}{\partial x_j}\right\} - \frac{\partial f}{\partial u}(x, \bar{u}, \nabla\bar{u}).$$

Similarly,

$$\begin{aligned} \mathscr{B}(x, u; D)w &= B(x, u + \bar{u}; D)w + \int_0^1 B_u(x, \theta u + \bar{u}; D)d\theta(w, \bar{u}) \\ &= \sum_{i,j=1}^{n} a_{ij}(x, \bar{u} + u)v_i(x)\frac{\partial w}{\partial x_j} + \tilde{g}(x, u)w \end{aligned}$$

is a boundary operator on $\partial\Omega$ with a function

$$\tilde{g}(x, u) = g(x, \bar{u} + u) + \int_0^1 \left\{\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial u}(x, \bar{u} + \theta u)v_i(x)\frac{\partial \bar{u}}{\partial x_j} + \frac{\partial g}{\partial u}(x, \bar{u} + \theta u)\bar{u}\right\}d\theta.$$

$\not{f}(x, u, \nabla u)$ is given by

$$\begin{aligned} (6.2)\qquad \not{f}(x, u, \nabla u) &= - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[\int_0^1 \left\{\frac{\partial a_{ij}}{\partial u}(x, \bar{u} + \theta u) - \frac{\partial a_{ij}}{\partial u}(x, \bar{u})\right\}d\theta u \frac{\partial \bar{u}}{\partial x_j}\right] \\ &\quad + \int_0^1 \left\{\frac{\partial f}{\partial u}(x, \bar{u} + \theta u, \nabla(\bar{u} + \theta u)) - \frac{\partial f}{\partial u}(x, \bar{u}, \nabla\bar{u})\right\}d\theta u \\ &\quad + \int_0^1 \{\nabla_q f(x, \bar{u} + \theta u, \nabla(\bar{u} + \theta u)) - \nabla_q f(x, \bar{u}, \nabla\bar{u})\}d\theta \cdot \nabla u. \end{aligned}$$

And, $u_0 = v_0 - \bar{u}$ is an initial function.

Then, the abstract formulation of (6.1) in the underlying spaces

$$X = L^p(\Omega), \quad Y_1 = W_p^{h-1}(\Omega), \quad Y_2 = W_p^1(\Omega), \quad Z = W_p^h(\Omega),$$

where $n < p < \infty$ and $1 + n/p < h < \mathrm{Min}\,\{a, 2\}$, is the following:

$$\text{(6.3)} \qquad \begin{cases} du/dt + A(u)u = f(u), & 0 < t < \infty, \\ u(0) = u_0. \end{cases}$$

Here, $A(u)$ denote linear operators determined by

$$\begin{cases} \mathscr{D}(A(u)) = \{w \in W_p^2(\Omega);\ \mathscr{B}(x, u; D)w = 0 \text{ on } \partial\Omega\} \\ A(u)w = \mathscr{A}(x, u; D)w \end{cases}$$

for $u \in K = \{u \in Z;\ \|u\|_Z < R\}$ with some $0 < R < \infty$. And, $f(u) = \not{f}(x, u, \nabla u)$ for $u \in K$.

In order to apply the Theorems 4.2 and 5.1, let us first verify that the Conditions (A.i, ii), (S.i, ii, iii, iv), (f.i, ii) and (Ex) (except (S.p) and (In)) are all fulfilled under the Assumptions (Ω), (a.1, 2), (f) and (g) announced above. But, as such verification has been already done in the previous paper [19, Sec. 5], we may here content ourselves with making a brief sketch only.

From (a.1) and (g), if $u \in K \subset \mathscr{C}^1(\bar{\Omega})$, then $a_{ij}(x, \bar{u} + u) \in \mathscr{C}^1(\bar{\Omega})$ and $\tilde{g}(x, u) \in \mathscr{C}^1(\partial\Omega)$; in addition, because of $\bar{u} \in \mathscr{C}^2(\bar{\Omega})$, b_i and $c \in \mathscr{C}(\bar{\Omega})$. Then it is known that the strong ellipticity (a.2) implies (A.i) with some $0 < \theta_0 < \pi/2$ and sufficiently large ω.

Let $u, v \in K$. According to the proof of [19, Proposition 5.1], we have:

$$\begin{aligned} &\|[A(u) - \omega]^{\nu_1}\{(\omega - A(u))^{-1} - (\omega - A(v))^{-1}\}\|_{\mathscr{L}(X)} \\ &\qquad \le C\{\sum_{i,j} \|a_{ij}(x, \bar{u} + u) - a_{ij}(x, \bar{u} + v)\|_{L^p(\Omega)} + \|\tilde{g}(x, u) - \tilde{g}(x, v)\|_{L^p(\partial\Omega)}\} \end{aligned}$$

for any $0 < \nu_1 < 1/2$. Similarly,

$$\begin{aligned} &\|[A(u) - \omega]^{\nu_2}\{(\omega - A(u))^{-1} - (\omega - A(v))^{-1}\}\|_{\mathscr{L}(X)} \\ &\qquad \le C\{\sum_{i,j} \|a_{ij}(x, \bar{u} + u) - a_{ij}(x, \bar{u} + v)\|_{W_p^1(\Omega)} + \|\tilde{g}(x, u) - \tilde{g}(x, v)\|_{L^p(\partial\Omega)}\} \end{aligned}$$

for any $0 < \nu_2 < (p + 1)/2p$. Then we easily verify that (A.ii) is valid with any $0 < \nu_1 < 1/2$ and any $0 < \nu_2 < (p + 1)/2p$.

It is known that (S.i) holds with $\gamma_1 = 1/h$ and with any $0 < \gamma_2 < (h - 1)/h$. As was shown in [19, Appendix], we can estimate the domains of the fractional powers $[A(u) - \omega]^\theta$, $0 \le \theta \le 1$; according to this, (S.ii) and (S.iii) are the case provided $h/2 < \alpha < 1$, $(h - 1)/2 < \alpha_1 < 1$ and $1/2 < \alpha_2 < 1$ respectively. Finally, (S.iv) is seen from the fact that the closed unit ball of a reflexive Banach space is sequentially weakly compact.

(f.i) and (f.ii) are verified directly from (6.2).

Finally, (Ex) is now evident from the above; especially, we can and in fact do take α so that $h/2 < \alpha < a/2$ (remember that a was an exponent for which $v_0 \in W_p^a(\Omega)$).

At this moment it is thus sufficient to verify (S.p) and (In). For this, however, we have to make further more essential assumptions.

(ū) The operator $A(0)$ determined by $\mathscr{A}(x, 0; D)$ and $\mathscr{B}(x, 0; D)$, more precisely by

$$\mathscr{A}(x, 0; D)w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left\{a_{ij}(x, \bar{u})\frac{\partial w}{\partial x_j}\right\} + \sum_{i=1}^{n} b_i(x)\frac{\partial w}{\partial x_i} + c(x)w$$

and

$$\mathscr{B}(x, 0; D)w = \sum_{i,j=1}^{n} a_{ij}(x, \bar{u})v_i(x)\frac{\partial w}{\partial x_j} + g(x, \bar{u})w + h(x)w$$

with

$$h(x) = \sum_{i,j=1}^{n} \frac{\partial a_{i,j}}{\partial u}(x, \bar{u})v_i(x)\frac{\partial \bar{u}}{\partial x_j} + \frac{\partial g}{\partial u}(x, \bar{u})\bar{u},$$

possesses the resolvent set containing a half plane $\{\lambda \in \boldsymbol{C}; \operatorname{Re}\lambda \leq \delta\}$ for some $\delta > 0$.

(v_0) The initial function $v_0 \in W_p^a(\Omega)$ $(a > h > 1 + n/p)$ lies in the ball: $\|v_0 - \bar{u}\|_Z < R$, and satisfies the compatibility condition:

$$\sum_{i,j=1}^{n} a_{ij}(x, v_0)v_i(x)\frac{\partial v_0}{\partial x_j} + g(x, v_0)v_0 = 0 \qquad \text{on } \partial\Omega. \tag{6.4}$$

We are now ready to apply the Theorems 4.2 and 5.1 and to state the main result of this section.

Theorem 6.1. *Under* (Ω), (a.1, 2), (f) *and* (g), *let the stationary solution* $\bar{u} \in \mathscr{C}^2(\bar{\Omega})$ *to* (D) *satisfy* (ū). *Then, there exists some neighborhood* V *of* $\bar{u}$ *in* $W_p^a(\Omega)$, *where* $n < p < \infty$ *and* $1 + n/p < a$, *such that, if the initial function* v_0 *lies in* V *and satisfies* (v_0), *then* (D) *possesses a global solution* v *such that, for any* $0 < \beta < \delta$,

$$\|v(t) - \bar{u}\|_{W_p^2(\Omega)} \leq C_\beta e^{-\beta t}\|v_0 - \bar{u}\|_{W_p^a(\Omega)}, \qquad 1 \leq t < \infty, \tag{6.5}$$

with some constant C_β *independent of* v_0.

Proof. The only thing to be noticed here is that (v_0) implies (In) for $u_0 = v_0 - \bar{u}$. Indeed, (6.4) means that $B(x, v_0; D)v_0 = 0$; so that, it follows that $\mathscr{B}(x, u_0; D)u_0 = 0$; then, since $a > 2\alpha$, (In) is verifed by [19, Theorem A.2]. Thus the Theorem 4.2 provides the global existence of solution to (6.3); obviously, $v = u + \bar{u}$ is a solution to (D). (6.5) follows from the Theorem 5.1 and from the well known estimate

$$\|w\|_{W_p^2(\Omega)} \leq C\{\|A(u)w\|_X + \|w\|_X\}, \qquad w \in \mathscr{D}(A(u)),\ w \in K.$$

References

[1] Amann, H., Parabolic evolution equations in interpolation and extrapolation spaces, J. Funct. Anal., **78** (1988), 233–270.

[2] Amann, H., Dynamic theory of quasilinear parabolic equations—I. Abstract evolution equations, Nonlinear Anal., **12** (1988), 895–919.

[3] Amann, H., Dynamic theory of quasilinear parabolic equations—II. Reaction-diffusion systems, Differential Integral Equations, **3** (1990), 13–75.

[4] Amann, H., Dynamic theory of quasilinear parabolic systems—III. Global existence, Math. Z., **202** (1989), 219–250.

[5] Drangeid, A. K., The principle of linearized stability for quasilinear parabolic evolution equations, Nonlinear Anal., **13** (1989), 1091–1113.

[6] Furuya, K., Analyticity of solutions of quasilinear evolution equations, Osaka J. Math., **18** (1981), 669–698.

[7] Furuya, K., Analyticity of solutions of quasilinear evolution equations II, Osaka J. Math., **20** (1983), 217–236.

[8] Hoshino, H. and Yamada, Y., Solvability and smoothing effect for semilinear parabolic equations, Funkcial. Ekvac., **34** (1991), 475–494.

[9] Kawanago, T., The behavior of solutions of quasi-linear hear eqations, Osaka J. Math., **27** (1990), 769–796.

[10] Kawanago, T., Large time behavior of solutions of quasi-linear heat conduction equations, preprint.

[11] Lunardi. A., Asymptotic exponential stability in quasilinear parabolic equations, Nonlinear Anal., **9** (1985), 563–586.

[12] Potier-Ferry, M., The linearization principle for the stability of solutions of quasilinear parabolic equations I, Arch. Rational Mech. Anal., **77** (1981), 301–320.

[13] Shigesada, N., Kawasaki, K. and Teramoto, E., Spatial segregation of interacting species, J. Theoret. Biol., **79** (1979), 83–99.

[14] Sobolevskii, P. E., Parabolic equations in Banach space with an unbounded variable operator, a fractional power of which has a constant domain of definition, Soviet Math. Dokl., **2** (1961), 545–548.

[15] Yagi, A., Fractional powers of operators and evolution equations of parabolic type, Proc. Japan Acad. Ser. A, **64** (1988), 227–230.

[16] Yagi, A., Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups, Funkcial. Ekvac., **32** (1989), 107–124.

[17] Yagi, A., Parabolic evolution equations in which the coefficients are the generators of intinitely differentiable semigroups II, Funkcial. Ekvac., **33** (1990), 139–150.

[18] Yagi, A., Abstract quasilinear evolution equations of parabolic type in Banach spaces,

Boll. Un. Mat. Ital. B(6) **5** (1991), 341–368.

[19] Yagi, A., Global solution to some quasilinear parabolic problem in mathematical biology, preprint.

[20] Yagi, A., Global solution to some quasilinear parabolic system in population dynamics, preprint.

[21] Tanabe, H., *Equations of Evolution*, Tokyo, Iwanami, 1975 (in Japanese). English translation, London, Pitman, 1979.

[22] Triebel, H., *Interpolation theory, Function spaces, Differential operators*, Amsterdam, North-Holland, 1987.

[23] Wahl, W. von, *The Equation of Navier-Stokes and Abstract Parabolic Equations*, Braunschweig/Wiesbaden, Vieweg, 1985.

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