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Periodic Solutions of Quasi-Linear Partial Functional Differential Equations with Unbounded Delay

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1 Introduction

Let X be a Banach space with norm $\|\cdot\|$ and suppose that $A: D(A) \to X$ is the infinitesimal generator of a strongly continuous operator semigroup $T(t)$ defined on X. The objective of this work is to study the existence of solutions and periodic solutions of a class of partial functional differential equations with unbounded delay. Let $\mathscr{B}$ be an abstract phase space. We will consider initial value problems which can be modelled as the abstract Cauchy problem:

$$\dot{x}(t) = Ax(t) + F(t, x_t), \qquad t \geq 0, \tag{1.1}$$

with initial condition

$$x_0 = \varphi \in \mathscr{B}, \tag{1.2}$$

where $F: (-\infty, a) \times \Omega \to X$, $a > 0$, is a continuous function and x_t represents the function defined from $(-\infty, 0]$ into X by $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leq 0$.

Similar problems have been studied by several authors. We only mention the works of Lightbourne [5], Travis and Webb [10] and recently, Shin [9]. In [5] the existence of periodic solutions of a non-delayed equation was established while in [10] it was obtained existence of solutions for an equation with finite delay and Shin [9] considers the problem of existence of solutions of equations where the function in the right member depends continuously on x_t.

Throughout this paper we will employ the phase space $\mathscr{B}$ introduced by Hale and Kato [3], but defined as in the book [4]. Thus, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$. We will assume that $\mathscr{B}$ satisfies the following axioms:

(A) If $x: (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathscr{B}$

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then for every t in $[\sigma, \sigma + a)$ the following conditions hold:

(i) x_t is in $\mathscr{B}$.

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathscr{B}}$.

(iii) $\| x_t \|_{\mathscr{B}} \leq K(t - \sigma) \sup \{ \| x(s) \| : \sigma \leq s \leq t \} + M(t - \sigma) \| x_\sigma \|_{\mathscr{B}}$.

Where $H \geq 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous and M is locally bounded and H, K and M are independent of x.

(A_1) For the function x in (A), x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathscr{B}$ is complete.

We will denote by $\hat{\mathscr{B}}$ the quotient Banach space $\mathscr{B}/\| \cdot \|_{\mathscr{B}}$. If E is a subset of $\mathscr{B}$ then we define $\hat{E} := \{ \hat{\varphi} : \varphi \in E \}$, where $\hat{\varphi}$ is the equivalence class of φ.

Furthermore we will reserve the symbol α to denote the Kuratowski's measure of non-compactness. Let B a bounded set of a seminormed space Y. The α-measure of B is defined as the infimum of $\varepsilon > 0$ such that B has a finite cover with sets of diameter less than ε. For the properties of the measure α see Deimling [1]. We will represent with the symbol $B_r[y]$ the closed ball centered at y and of radius $r \geq 0$.

For the theory of strongly continuous semigroups we refer to Nagel [7]. In particular, it is well known that there exist constants $\tilde{M} \geq 1$ and $\omega \in \boldsymbol{R}$ such that

$$\| T(t) \| \leq \tilde{M} e^{\omega t}, \qquad t \geq 0. \tag{1.3}$$

In section 2 we will establish a result of existence of solutions and some of their properties while in section 3 we will study the existence of periodic solutions. In the sequel we will assume that $\mathscr{B}$ satisfies axioms (A), (A_1) and (B).

2 Existence of solutions

Henceforth we will assume that $F : [0, a) \times \Omega \to X$ is a continuous function, where Ω is a nonempty open subset of $\mathscr{B}$.

We will say that a function $x : (-\infty, b) \to X$, $0 < b < a$, is a mild solution of the Cauchy problem (1.1)–(1.2) if $x_0 = \varphi$ and the restriction $x : [0, b) \to X$ is continuous and satisfies the integral equation:

$$x(t) = T(t)\varphi(0) + \int_0^t T(t - s) F(s, x_s) ds, \qquad 0 \leq t < b. \tag{2.1}$$

In the rest of this work we will abbreviate our terminology calling solutions to the mild solutions.

In this section we will use the Schauder's fixed point theorem to obtain the existence of solutions. In order to apply the Schauder's theorem we will introduce a compactness condition on the composition of the semigroup $T(t)$ and the function F.

Theorem 1 *Let $\varphi \in \mathscr{B}$. Assume that there exist positive constants b_φ and r_φ such that $B_{r_\varphi}[\varphi] \subseteq \Omega$ and for each $0 < t \leq b_\varphi$ there is a compact set $W_t \subset X$ such that $T(t)F(s, \psi) \in W_t$, for every $\psi \in B_{r_\varphi}[\varphi]$ and all $0 \leq s \leq b_\varphi$. Then there exists a solution of* (1.1)–(1.2) *defined on $(-\infty, b)$, for some $0 < b < b_\varphi$.*

Proof. From the properties of strongly continuous semigroups we may assume that $\| T(t) \| \leq \tilde{M}_a$, for $0 \leq t \leq a$ and for some constant $\tilde{M}_a \geq 1$. Since F is a continuous function, there exist constants $0 < b_1 \leq b_\varphi$, $0 < r_1 \leq r_\varphi$ and $N \geq 0$ such that $\| F(s, \psi) \| \leq N$, for every $0 \leq s \leq b_1$ and all $\psi \in B_{r_1}[\varphi]$.

Let us define the function $y(t) = T(t)\varphi(0)$ for $t \geq 0$ and $y_0 = \varphi$. Then from the properties of $\mathscr{B}$ we infer that $y_t \in \mathscr{B}$ and that, for every $0 < r_2 < r_1$, there exists b_2, $0 < b_2 \leq b_1$, such that $\| y_t - \varphi \|_{\mathscr{B}} \leq r_2$, for all $0 \leq t \leq b_2$. We set $K_a := \max\{K(t) : 0 \leq t \leq a\}$. Let b be any constant such that $0 < b < \min\{b_2, (r_1 - r_2)/\tilde{M}_a K_a N\}$. Let us introduce the space $\mathscr{F}_b$ of all functions $u : (-\infty, b] \to X$ such that $u_0 \in \mathscr{B}$ and the restriction $u : [0, b] \to X$ is continuous and let $\| \cdot \|_{\mathscr{F}}$ be a seminorm in $\mathscr{F}_b$ defined by

$$\| u \|_{\mathscr{F}} = \| u_0 \|_{\mathscr{B}} + \sup\{\| u(s) \| : 0 \leq s \leq b\}.$$

We write $u \in \mathscr{F}_b(\varphi)$ if $u \in \mathscr{F}_b$, $\| u_0 - \varphi \|_{\mathscr{B}} = 0$ and $\| u_t - \varphi \|_{\mathscr{B}} \leq r_1$, for every $0 \leq t \leq b$.

Clearly $\mathscr{F}_b(\varphi)$ is a non empty, bounded, convex and closed subset of $\mathscr{F}_b$. The first assertion follows from $y \in \mathscr{F}_b(\varphi)$. In order to show that $\mathscr{F}_b(\varphi)$ is bounded it is sufficient to observe that from axiom (Aii) follows that

$$\begin{aligned} \| u(t) \| &\leq H \| u_t \|_{\mathscr{B}} \leq H \| u_t - \varphi \|_{\mathscr{B}} + H \cdot \| \varphi \|_{\mathscr{B}} \\ &\leq H(r_1 + \| \varphi \|), \end{aligned}$$

for all $0 \leq t \leq b$. To prove $\mathscr{F}_b(\varphi)$ is closed, we will consider a sequence $(u^n)_n$ in $\mathscr{F}_b(\varphi)$, which converges to $u \in \mathscr{F}_b$. Then $\| u_0 - \varphi \|_{\mathscr{B}} = 0$ and, it follows from axiom (Aiii) that

$$\| u_t^n - u_t \|_{\mathscr{B}} \leq K(t) \sup_{0 \leq s \leq t} \| u^n(s) - u(s) \| + M(t) \| u_0^n - u_0 \|.$$

Thus $u_t^n \to u_t$, $n \to \infty$, for $0 \leq t \leq b$, which in turn proves that $\| u_t - \varphi \|_{\mathscr{B}} \leq r_1$. The convexity of $\mathscr{F}_b(\varphi)$ is an immediate consequence of definitions.

Now we define the map $\mathscr{T}$ on $\mathscr{F}_b(\varphi)$ by the expression:

$$(\mathscr{T}u)(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s, u_s)ds, \tag{2.2}$$

for all $0 \le t \le b$, and $(\mathscr{T}u)_0 = \varphi$: It will be shown that $\mathscr{T}$ is a continuous map from $\mathscr{F}_b(\varphi)$ into $\mathscr{F}_b(\varphi)$. It is clear that $v = \mathscr{T}u$ is continuous on $[0, b]$ so that $v \in \mathscr{F}_b$. If we set $w = v - y$, then

$$\begin{aligned} \| v_t - \varphi \|_{\mathscr{B}} &\le \| y_t - \varphi \|_{\mathscr{B}} + \| w_t \|_{\mathscr{B}} \\ &\le r_2 + \| w_t \|_{\mathscr{B}}. \end{aligned}$$

From axiom (Aiii) we obtain that

$$\| w_t \|_{\mathscr{B}} \le K(t) \sup \{ \| w(s) \| : 0 \le s \le t \}$$

and since

$$\begin{aligned} \| w(t) \| &\le \int_0^t \tilde{M}_a \cdot \| F(s, u_s) \| \, ds \\ &\le \tilde{M}_a N b \\ &\le \frac{1}{K_a} (r_1 - r_2) \end{aligned}$$

we may conclude that

$$\| v_t - \varphi \|_{\mathscr{B}} \le r_1, \qquad 0 \le t \le b,$$

which shows that $v \in \mathscr{F}_b(\varphi)$.

On the other hand, from axiom (A_1) we obtain immediately that the map $[0, b] \times \mathscr{F}_b \to \mathscr{B}$, $(s, u) \to u_s$, is continuous. If $(u^n)_{n \ge 1}$ is a convergent sequence in $\mathscr{F}_b(\varphi)$ with limit u, then the set $\{u, u^n : n \ge 1\}$ is compact in $\mathscr{F}_b$. Hence the set $W = \{(s, u_s), (s, u_s^n) : n \ge 1, 0 \le s \le b\}$ is compact in $[0, b] \times \mathscr{B}$ and the function F is uniformly continous on W. Since $(u^n)_n$ converges to u, we conclude from (2.2) that $(\mathscr{T}u^n)_n$ converges to $\mathscr{T}u$, which proves that $\mathscr{T}$ is continuous.

Next we will prove that the range of $\mathscr{T}$ is relatively compact. By Ascoli's theorem it is sufficient to show that the set $\mathscr{R}(\mathscr{T}) = \{\mathscr{T}u : u \in \mathscr{F}_b(\varphi)\}$ is equicontinuous on $[0, b]$ and $\mathscr{R}(\mathscr{T})(t)$ is relatively compact in X for each $0 \le t \le b$. To prove the first assertion we consider $0 < t_0 < t \le b$ and $0 < \varepsilon < t_0$. From the definition of $\mathscr{T}$ it follows that

$$\begin{aligned}\mathscr{T}(u)(t)-\mathscr{T}(u)(t_0) &= [T(t)-T(t_0)]\varphi(0)\\ &+\int_0^{t_0-\varepsilon}[T(t-s)-T(t_0-s)]F(s,u_s)ds\\ &+\int_{t_0-\varepsilon}^{t_0}[T(t-s)-T(t_0-s)]F(s,u_s)ds\\ &+\int_{t_0}^{t}T(t-s)F(s,u_s)ds\end{aligned}$$

and this equality yields

(2.3)

$$\begin{aligned}\|\mathscr{T}(u)(t)-\mathscr{T}(u)(t_0)\| &\le \|[T(t)-T(t_0)]\varphi(0)\|+\tilde{M}_aN(t-t_0)+2\tilde{M}_aN\varepsilon\\ &+\left\|\int_0^{t_0-\varepsilon}[T(t-\varepsilon-s)-T(t_0-\varepsilon-s)]T(\varepsilon)F(s,u_s)ds\right\|.\end{aligned}$$

Besides $T(\varepsilon)F(s, u_s)$ is included in a compact set W_ε, for all $0 \le s \le b$ and all $u \in \mathscr{F}_b(\varphi)$. Since the functions $T(\cdot)x$, $x \in W_\varepsilon$, are equicontinuous, there exists $\delta > 0$ such that

$$\|T(t_1)x-T(t_2)x\| \le \varepsilon, \qquad x \in W_\varepsilon, \tag{2.4}$$

when $|t_1 - t_2| \le \delta$. Consequently, if $|t - t_0| \le \delta$, substituting (2.4) into (2.3) we obtain

$$\begin{aligned}\|\mathscr{T}(u)(t)-\mathscr{T}(u)(t_0)\| &\le \|[T(t)-T(t_0)]\varphi(0)\|\\ &+\tilde{M}_aN(t-t_0)+2\tilde{M}_aN\varepsilon+\varepsilon b\end{aligned}$$

which shows that $\mathscr{R}(\mathscr{T})$ is equicontinuous from the right at t_0. In the same manner, we can prove that $\mathscr{R}(\mathscr{T})$ is equicontinuous at any $t_0 \ge 0$. We omit the details.

Now we will prove that $\mathscr{R}(\mathscr{T})(t)$ is a relatively compact set for each $0 \le t \le b$. Since $T(t)\varphi(0)$ does not depend on $u \in \mathscr{F}_b(\varphi)$, it is sufficient to prove that the set of all vectors $(\mathscr{T}u)(t) - T(t)\varphi(0)$ is relatively compact.

Clearly we may suppose that $t > 0$. Let $0 < \varepsilon < t$.

Then

$$\begin{aligned}\int_0^t T(t-s)F(s,u_s)ds &= \int_0^{t-\varepsilon}T(t-s-\varepsilon)T(\varepsilon)F(s,u_s)ds\\ &+\int_{t-\varepsilon}^{t}T(t-s)F(s,u_s)ds.\end{aligned}$$

Let W_ε be a compact set such that $T(\varepsilon)F(s, u_s) \in W_\varepsilon$, for every $u \in \mathscr{F}_b(\varphi)$ and all $0 \leq s \leq t$. Then the set $V_\varepsilon := \{T(s)w : 0 \leq s \leq t - \varepsilon,\ w \in W_\varepsilon\}$ is compact. By the mean value theorem for the Bochner integral ([6]) we may infer that

$$\int_0^{t-\varepsilon} T(t-s-\varepsilon)T(\varepsilon)F(s, u_s)ds \in (t-\varepsilon)\overline{c(V_\varepsilon)},$$

where $c(V_\varepsilon)$ denotes the convex hull of V_ε. Since $\overline{c(V_\varepsilon)}$ is compact (see [2], Theorem V. 2.6) and

$$\left\| \int_{t-\varepsilon}^{t} T(t-s)F(s, u_s)ds \right\| \leq \tilde{M}_a N \varepsilon, \qquad u \in \mathscr{F}_b(\varphi),$$

we conclude that $\mathscr{R}(\mathscr{T})(t)$ is relatively compact.

Finally, the Schauder's fixed point theorem asserts that $\mathscr{T}$ has a fixed point in $\mathscr{F}_b(\varphi)$, which is a solution of (1.1)–(1.2).

So far, we have only considered the abstract Cauchy problem (1.1)–(1.2) with initial condition at $t = 0$. Nevertheless, the same argument used in the previous theorem allows us to establish the existence of local solutions of the problem

$$\dot{x}(t) = Ax(t) + F(t, x_t), \qquad t \geq \sigma, \tag{2.5}$$

$$x_\sigma = \varphi, \tag{2.6}$$

where $0 \leq \sigma < a$.

Proposition 1 *Suppose for every closed and bounded set $B \subseteq \Omega$ and every $0 \leq \sigma < a$, there is $b' > 0$ such that, for all $0 < t \leq b'$ there exists a compact set W_t such that $T(t)F(s, \psi) \in W_t$, for every $\psi \in B$ and all $\sigma \leq s \leq \sigma + b' < a$. Then for every $\varphi \in \Omega$ and $0 \leq \sigma < a$ there exists a solution of* (2.5)–(2.6) *defined on $(-\infty, \sigma + b)$, for some $0 < b \leq b'$.*

Related with this result, it should be noted that it can be established, in the usual form, that if F satisfies a local Lipschitz condition then the solution of (2.5)–(2.6) is unique. Furthermore, under the conditions of the preceding proposition, for every $0 < \sigma < a$ and every $\varphi \in \Omega$ there exists a maximal or noncontinuable solution defined on an interval $(-\infty, \sigma + b)$.

Hereafter we will be interested in the existence of global solutions. Our next two propositions are extensions of well known results [3].

Proposition 2 *Assume F satisfies the following condition: For every closed and bounded set $B \subseteq \Omega$ and every $t > 0$ there exists a compact set W_t such that $T(t)F(s, \psi) \in W_t$ for every $\psi \in B$ and all $0 \leq s < a$. If $x : (-\infty, \sigma + b) \to X$ is a maximal solution of* (2.5)–(2.6) *and W is a compact subset of $[0, a) \times \Omega$ then* *there exists t_W such that $(t, x_t) \notin W$, for $t_W \leq t < \sigma + b$.*

We will omit the proof because the main part of the argument used in the demonstration of Theorem 2.3 in [3] also serves in this case.

Corollary 1 *Let $F: [0, \infty) \times \mathscr{B} \to X$ be a continuous function which satisfies the following condition:*
(F_1). *For every closed and bounded set $B \subseteq \Omega$, and every $a, t > 0$ there exists a compact set W_t such that $T(t)F(s, \psi) \in W_t$, for all $\psi \in B$ and all $0 \leq s \leq a$.*

If $x: (-\infty, b) \to X$, $b > 0$, is a maximal solution of problem (1.1)–(1.2) *and $\{x_t: 0 \leq t < b\}$ is relatively compact in $\mathscr{B}$, then $b = \infty$.*

Proof. It is an immediate consequence of Proposition 2.

Proposition 3 *Suppose F satisfies the condition of Proposition* 2 *and takes bounded and closed sets into bounded sets. If $x: (-\infty, b) \to X$ is a maximal solution of* (1.1)–(1.2) *and U is a closed and bounded set included in $[0, a) \times \Omega$ then there exists a sequence $t_k \to b^-$ such that $(t_k, x_{t_k}) \notin U$. Further, if F is defined on $[0, \infty) \times \mathscr{B}$ then there is a t_U such that $(t, x_t) \notin U$, for $t_U \leq t < b$.*

Proof. Let us assume that $(t, x_t) \in U$, for $0 < t_0 \leq t < b$. By the hypotheses we may write

$$\| F(t, x_t) \| \leq N, \qquad 0 \leq t < b,$$

for some positive constant N. Now we will prove that the restriction $x: [t_0, b) \to X$ is uniformly continuous. In fact, if $t_0 \leq t \leq t' < b$ and $0 < \varepsilon < t_0$, we can write

$$x(t') - x(t) = T(t')\varphi(0) - T(t)\varphi(0) + \int_0^{t-\varepsilon} [T(t'-s) - T(t-s)]F(s, x_s)ds$$
$$+ \int_{t-\varepsilon}^{t} [T(t'-s) - T(t-s)]F(s, x_s)ds + \int_t^{t'} T(t'-s)F(s, x)ds$$

and evaluating the right member as in the proof of Theorem 1 we can conclude that $\| x(t') - x(t) \|$ tends to zero uniformly as $|t - t'|$ tends to zero.

Therefore, the set $\{x(t): t_0 \leq t < b\}$ is totally bounded in X. From these properties we infer that there exists $\lim_{t \to b^-} x(t) = z$. If we define $x(b) = z$, then the function $x(\cdot)$ is continuous in $[0, b]$ and the axiom (A_1) implies that $x_t \to x_b$, as $t \to b^-$. Since $(b, x_b) \in [0, a) \times \Omega$, from Proposition 1 we obtain the existence of a solution of a problem (2.5)–(2.6) with $\sigma = b$ and $\varphi = x_b$, which is contrary to our assumption that x is a maximal solution. Consequently, there exists a monotonically increasing sequence $(t_k)_k$ which converges to b such that $(t_k, x_{t_k}) \notin U$.

The second assertion can be proved arguing in the same manner as in the proof of Theorem 2.4 in [3].

Corollary 2 *Let* $F: [0, \infty) \times \mathscr{B} \to X$ *be a continuous function which satisfies* (F_1) *and takes bounded and closed sets into bounded sets. If* $x: (-\infty, b) \to X$ *is a maximal solution of* (1.1)–(1.2) *and the set* $\{x_t: 0 \le t < b\}$ *is bounded, then* $b = +\infty$.

Proof. Let us assume that $b < \infty$. We choose $a > b$ and we put $\Gamma = \overline{\{x_t: 0 \le t < b\}}$ and $U = \{(t, \psi): 0 \le t \le b, \psi \in \Gamma\}$. Then U is a closed and bounded set included in Ω and $(t, x_t) \in U$ for $0 \le t < b$. This is a contradiction to Proposition 3. □

In the sequel we will assume that the solution $x(\cdot, \varphi)$ of problem (1.1)–(1.2) for all φ belonging to some subset E of $\mathscr{B}$ is unique and defined on $[0, \infty)$. For $\tau > 0$, we consider the map $P: E \to \mathscr{B}$, defined by $P(\varphi) := x_\tau(\cdot, \varphi)$. Our results about periodic solutions depend on the continuity of P.

Proposition 4 *Suppose* $F: [0, \infty) \times \Omega \to X$ *is a continuous function which takes bounded sets into bounded sets and satisfies the condition* (F_1). *If for each* $\varphi \in E$ *there exists* $r_\varphi > 0$ *such that* $\{x_t(\cdot, \psi): 0 \le t \le \tau, \psi \in B_{r_\varphi}[\varphi]\}$ *is included in a bounded and closed subset of* Ω, *then* P *is a continuous map.*

Proof. Let us fix $\varphi \in E$ and let $(\varphi^n)_n$ be a sequence in E convergent to φ. Clearly we may assume that $\|\varphi^n - \varphi\|_{\mathscr{B}} \le r_\varphi$, for all $n \in \boldsymbol{N}$. Moreover, the set $\{\varphi^n(0): n \in \boldsymbol{N}\}$ is compact in X. Proceeding as in Theorem 1 we can prove that the set $\{x(\cdot, \varphi^n): n \in \boldsymbol{N}\}$ is relatively compact in $C([0, \tau]; X)$. If $(\psi^n)_n$ is a subsequence of $(\varphi^n)_n$, then there exists a subsequence $(x(\cdot, \psi^{n_k}))_k$ of $(x(\cdot, \psi^n))_n$ which converges to some function $u(\cdot) \in C([0, \tau]; X)$. Next, we also represent by $u(\cdot)$ the extension defined by $u(\theta) := \varphi(\theta)$, for $\theta < 0$. From our hypotheses and the axioms of space $\mathscr{B}$, we obtain that $x_s(\cdot, \psi^{n_k}) \to u_s$, as $k \to \infty$, and that $u_s \in \Omega$, for all $0 \le s \le \tau$. Since $\{x_s(\cdot, \psi^{n_k}): 0 \le s \le \tau, k \in \boldsymbol{N}\}$ is a bounded subset of Ω and F takes bounded sets into bounded sets, using the Lebesgue's dominated convergence Theorem for the integration in the sense of Bochner ([6]), we obtain that $u(\cdot) = x(\cdot, \varphi)$, which shows that $P\psi^{n_k} \to P\varphi$, as $k \to \infty$. Since $(\psi^n)_n$ was an arbitrary subsequence of $(\varphi^n)_n$, this proves that P is a continuous map.

Essential for our results on periodic solutions it will be the existence of a bounded, closed and convex set $E \subseteq \mathscr{B}$ such that $P(E) \subseteq E$. In the next proposition we present a case where this occurs. In this result we will consider as phase space $\mathscr{B} := C \times L^1(g)$ the space of functions $\varphi: (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, for some $r > 0$, Lebesgue-measurable and $g \cdot \varphi$ is Lebesgue integrable on $(-\infty, -r)$, where $g: (-\infty, -r) \to \boldsymbol{R}$ is a positive Lebesgue integrable function. The seminorm in $\mathscr{B}$ is defined by

$$\|\varphi\| := \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0\} + \int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|\,d\theta.$$

We will assume that g satisfies conditions $(g-5)$ and $(g-6)$, in the terminology of [4]. This means that g is locally integrable on $(-\infty, -r)$ and that there exists a nonnegative and locally bounded function G on $(-\infty, -0]$ such that

$$g(\xi+\theta) \le G(\xi)g(\theta),$$

for all $\xi \le 0$ and $\theta \in (-\infty, -r)\setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure 0. Therefore, $\mathscr{B}$ is a phase space which verifies axioms (A), (A_1) and (B). Further, we will suppose that

$$\gamma(t) := \sup_{\theta \le -r} \frac{g(\theta - t)}{g(\theta)} + \gamma_0 + \int_{-t-r}^{-\bar{t}} g(\theta)d\theta \longrightarrow 0, \qquad \text{as } t \longrightarrow \infty,$$

where $\bar{t} = \max\{r, t\}$ and $\gamma_0 := 1$, when $t \le r$ and $\gamma_0 := 0$, when $t > r$.

For example, a function $g(\theta) = \exp(\mu\theta)$, $\mu > 0$, satisfies all the above conditions, as well as the conditions imposed in the next result, at least for μ large enough.

Proposition 5 *Let $F: [0, \infty) \times \mathscr{B} \to X$ be a continuous function which satisfies (F_1), a local Lipschitz condition and*

$$\|F(t, \varphi)\| \le N_1\|\varphi\| + N_2, \qquad \varphi \in \mathscr{B}, \tag{2.7}$$

for some constants $N_1, N_2 \ge 0$. If there exist $\omega > 0$ and $K_0 > 0$ for which the following conditions hold:

a) $\quad \|T(t)\| \le \tilde{M}e^{-\omega t}, \qquad t \ge 0,$

b) $\quad \displaystyle\int_0^{t-r} g(s-t)e^{\omega(t-r-s)}ds \le K_0,$

and

c) $\quad \tilde{M}N_1 e^{\omega r}(1 + K_0) < \omega.$

Then there exist $R > 0$ and $\tau > 0$ such that $P(B_R[0]) \subseteq B_R[0]$ and the set $\{x_t(\cdot, \varphi): \varphi \in B_R[0],\ 0 \le t \le \tau\}$ is bounded in $\mathscr{B}$.

Proof First we will prove that the solution $x(\cdot, \varphi)$ is defined on $[0, \infty)$ for all $\varphi \in \mathscr{B}$. In fact, if we use the abbreviated notation $x(\cdot)$ instead of $x(\cdot, \varphi)$ then from (2.1) we obtain

$$\|x(t)\| \leq \tilde{M}e^{-\omega t}\|\varphi(0)\| + \int_0^t \tilde{M}e^{-\omega(t-s)}(N_1\|x_s\| + N_2)ds \tag{2.8}$$

$$\leq \tilde{M}e^{-\omega t}\|\varphi\| + \frac{\tilde{M}N_2}{\omega} + \tilde{M}N_1\int_0^t e^{-\omega(t-s)}\|x_s\|ds.$$

On the other hand, the definition of seminorm in $\mathscr{B}$ yields

$$\|x_t\| = \sup_{-r\leq\theta\leq 0}\|x(t+\theta)\| + \int_{-\infty}^{-r} g(\theta)\|x(t+\theta)\|d\theta.$$

If $t \leq r$, then

$$\|x_t\| = \sup_{0\leq s\leq t}\|x(s)\| + \sup_{-r\leq\theta\leq -t}\|\varphi(t+\theta)\| + \int_{-\infty}^{-r} g(\theta)\|\varphi(t+\theta)\|d\theta$$

$$\leq \tilde{M}\|\varphi\| + \frac{\tilde{M}N_2}{\omega} + \tilde{M}N_1\int_0^t e^{\omega s}\|x_s\|ds + \gamma(t)\|\varphi\| \tag{2.9}$$

$$\leq \tilde{M}e^{-\omega(t-r)}\|\varphi\| + \gamma(t)\|\varphi\| + \frac{\tilde{M}N_2}{\omega} + \tilde{M}N_1e^{\omega r}\int_0^t e^{-\omega(t-s)}\|x_s\|ds$$

whereas, when $t \geq r$,

$$\|x_t\| \leq \sup_{t-r\leq s\leq t}\|x(s)\| + \int_{-\infty}^{-t} g(\theta)\|\varphi(t+\theta)\|d\theta + \int_{-t}^{-r} g(\theta)\|x(t+\theta)\|d\theta$$

and substituing (2.8) into the above expression we obtain

(2.10)

$$\|x_t\| \leq \tilde{M}e^{-\omega(t-r)}\|\varphi\| + \frac{\tilde{M}N_2}{\omega} + \tilde{M}N_1e^{\omega r}\int_0^t e^{-\omega(t-s)}\|x_s\|ds + \gamma(t)\|\varphi\|$$

$$+ \int_0^{t-r} g(s-t)\left[\tilde{M}e^{-\omega s}\|\varphi\| + \frac{\tilde{M}N_2}{\omega} + \tilde{M}N_1e^{\omega r}\int_0^s e^{-\omega(s-\xi)}\|x_\xi\|d\xi\right]ds.$$

Hence, from (2.9), (2.10) and Fubini's theorem we infer that

$$\|x_t\| \leq (\tilde{M}e^{-\omega(t-r)} + \gamma(t) + \tilde{M}K_0e^{-\omega(t-r)})\|\varphi\| + \frac{\tilde{M}N_2}{\omega}(1+K_0)$$

$$+ \tilde{M}N_1e^{\omega r}(1+K_0)\int_0^t e^{-\omega(t-s)}\|x_s\|ds,$$

for each $t \geq 0$. Now, a simple calculation using the Gronwall's lemma shows that

$$\|x_t\| \leq \tilde{M}(1 + K_0)e^{\omega r}(e^{-\omega t} + e^{-(\omega-v)t})\|\varphi\| + \gamma(t)\|\varphi\|$$
$$+ v \cdot \|\varphi\| \int_0^t e^{-\omega(t-s)}\gamma(s)ds + \frac{\tilde{M}N_2}{\omega}(1 + K_0)(1 + e^{-(\omega-v)t})$$

where $v = \tilde{M}N_1 e^{\omega r}(1 + K_0)$. Consequently, we can write

$$\|x_t\| \leq C_t\|\varphi\| + C,$$

where C is a constant and $C_t \to 0$, as $t \to \infty$. Then Corollary 2 implies that $x(\cdot)$ is defined on $[0, \infty)$. Furthermore, if we choose $R \geq 2C$ and $\tau > 0$ such that $C_\tau \leq 1/2$, we obtain the assertion.

For example, if we choose $g(\theta) = e^{\mu\theta}$, with $\mu > \omega$ then we can take $K_0 = e^{-\mu r}/(\mu - \omega)$ and the condition c) is obained if

$$r\tilde{M}N_1\left(1 + \frac{e^{-\mu r}}{\mu - \omega}\right) < 1/e.$$

□

3 Existence of periodic solutions

Let $\tau > 0$ and suppose that the function $F\colon [0, \infty) \times \Omega \to X$ is τ-periodic in t.

In this section, we consider the following condition:

(F_2). There exists a closed, bounded and convex subset $E \subseteq \Omega$ such that for all $\varphi \in E$ there exists a unique solution $x(\cdot, \varphi)$ of problem (1.1)–(1.2) defined on $[0, \infty)$ such that $x_\tau(\cdot, \varphi) \in E$ and the set $\{x_t(\cdot, \varphi)\colon 0 \leq t \leq \tau, \varphi \in E\}$ is included in a bounded and closed set in Ω.

We will represent by P the map defined on E by $P\varphi = x_\tau(\cdot, \varphi)$.

Let $x = x(\cdot\,\varphi)$ be the solution of (1.1)–(1.2) with initial condition $x_0 = \varphi$. We observe that $x_\tau = \varphi$ implies that x is a τ-periodic function. In fact, if we define $y(t) = x(t + \tau)$, for $t \in \boldsymbol{R}$, then for each $t \geq 0$,

$$\begin{aligned} y(t) &= T(t + \tau)\varphi(0) + \int_0^{t+\tau} T(t + \tau - s)F(s, x_s)ds \\ &= T(t)\left[T(\tau)\varphi(0) + \int_0^\tau T(\tau - s)F(s, x_s)ds\right] \\ &\quad + \int_0^t T(t - s)F(s + \tau, x_{s+\tau})ds \\ &= T(t)x(\tau) + \int_0^t T(t - s)F(s, y_s)ds, \end{aligned}$$

which yields that $y(\cdot)$ is a solution of equation (1.1) and since $y_0 = x_\tau = \varphi$, by the uniqueness of solution, we obtain that $y(t) = x(t+\tau) = x(t)$, for all $t \geq 0$. Therefore, in order to obtain a periosic solution it is sufficient to prove the existence of a fixed point of the map P and for this purpose we will employ the Sadovski's fixed point theorem ([8]) in the seminormed space $\mathscr{B}$.

Let Z be a Banach space and $B_0 \subset Z$. We recall that a continuous function $f\colon B_0 \to Z$ is said to be condensing if $\alpha(f(B)) < \alpha(B)$, for each bounded set $B \subset B_0$ with $\alpha(B) > 0$. If B_0 is a closed, bounded and convex set and $f\colon B_0 \to B_0$ is condensing then from the Sadovski's theorem we may assert that $\{z \in Z : f(z) = z\}$ is non-empty and compact.

Now we are in conditions to establish our main result.

Theorem 2 *Assume that T is a compact semigroup and F is a τ-periodic function which satisfies (F_2) and takes bounded subsets of $[0, \infty) \times E$ into bounded subsets of X. Moreover, suppose $\mathscr{B}$ satisfies axioms (A), (A_1) and (B), and the following condition:*

$$\inf_{0<\sigma<\tau} M(\tau-\sigma)[HK(\sigma) \sup_{0 \leq t \leq \sigma} \|T(t)\| + M(\sigma)] < 1. \tag{3.1}$$

Then the set of $\varphi \in E$ such that the solution $x(\cdot, \varphi)$ is τ-periodic is non-empty and compact.

Proof. Since T is compact then F satisfies condition (F_1) and by Proposition 4, P is a continuous map. Therefore, there exists an induced map $\hat{P}\colon \hat{E} \to \hat{E}$ which satisfies the condition $\hat{P}\hat{\varphi} = P(\varphi)$, for all $\hat{\varphi} \in E$ and every $\varphi \in \hat{\varphi}$. We will prove that $\hat{P}$ is condensing. It is clear that for every subset C of $\hat{E}$ there exists $D \subset E$ such that $C = \hat{D}$ and $\alpha(C) = \alpha(D)$. Consequently we may restrict us to consider a subset $\hat{D}$ of $\hat{E}$ with $\alpha(D) > 0$. If we denote by $D[\sigma, \tau]$, $0 \leq \sigma \leq \tau$, the set defined as

$$D[\sigma, \tau] := \{x(\cdot, \varphi)|_{[\sigma,\tau]} : \varphi \in D\}$$

then from Theorem 2.1 in [9] we obtain that

$$\alpha(\hat{P}(\hat{D})) \leq K(\tau-\sigma)\alpha(D[\sigma, \tau]) + M(\tau-\sigma)\alpha(\hat{D}_\sigma), \tag{3.2}$$

where $D_\sigma = \{x_\sigma(\cdot, \varphi) : \varphi \in D\}$.

On the other hand, for every $\sigma > 0$, the set $D[\sigma, \tau]$ is relatively compact in $C([\sigma, \tau]; X)$. To prove this assertion we need to show that $D[\sigma, \tau]$ is equicondinuous and each orbit $D[\sigma, \tau](t)$ is relatively compact in X, for $\sigma \leq t \leq \tau$. But the proof of both statements can be carried out with the same argument already used in the proof of Theorem 1. Of course, the compactness of semigroup T is now essential because we can write

$$D[\sigma, \tau] \subseteq \{T(\cdot)\varphi(0): \varphi \in D\} + \{w(\cdot, \varphi): \varphi \in D\}$$

where

$$w(t, \varphi) := \int_0^t T(t-s)F(s, x_s)ds, \tag{3.3}$$

and the first set in the right member of the above inclusion is relatively compact in view of $\{\varphi(0): \varphi \in D\}$ is bounded in X and $T(t)$ is a compact linear map for every $t \in [\sigma, \tau]$.

Combining this fact with (3.2) we see that

$$\alpha(\hat{P}(\hat{D})) \leq M(\tau - \sigma)\alpha(\hat{D}_\sigma). \tag{3.4}$$

Next we will evaluate $\alpha(\hat{D}_\sigma)$. Proceeding as above for the interval $[0, \sigma]$ we infer that

$$\alpha(\hat{D}_\sigma) \leq K(\sigma)\alpha(D[0, \sigma]) + M(\sigma)\alpha(\hat{D}). \tag{3.5}$$

We shall now show that

$$\alpha(D[0, \sigma]) \leq H \sup_{0 \leq t \leq \sigma} \| T(t) \| \alpha(D). \tag{3.6}$$

In fact, if $D(0) = \{\varphi(0): \varphi \in D\}$ then from the axiom (Aii) it follows that

$$\alpha(D(0)) \leq H\alpha(D). \tag{3.7}$$

Further, if we choose a real number $d > 0$ such that $\alpha(D(0)) < d$ then, from the definition of α, we can get a finite cover of $D(0)$ with sets $R_i \subseteq X$, $i = 1, 2, \cdots, n$, of diameter less than d. Next, for a subset R of X we will indicate by R^* the set of continuous functions

$$R^* = \{T(\cdot)x|_{[0,\sigma]}: x \in R\}.$$

Then, it is clear that

$$\operatorname{diam}(R_i^*) < \sup_{0 \leq t \leq \sigma} \| T(t) \| d$$

and

$$D(0)^* \subseteq \bigcup_{i=1}^{n} R_i^*.$$

Hence we obtain that

$$\alpha(D(0)^*) \leq \sup_{0 \leq t \leq \sigma} \| T(t) \| \cdot d.$$

Combining this inequality, (3.7) and the choice of d we obtain

$$\alpha(D(0)^*) \leq H \sup_{0 \leq t \leq \sigma} \| T(t) \| \cdot \alpha(D).$$

Since

$$D([0, \sigma]) \subseteq D(0)^* + \{w(\cdot, \varphi)|_{[0,\sigma]} : \varphi \in D\}$$

and the second set in the right member is relatively compact in $C([0, \sigma]; X)$ then from the properties of measure α, we complete the proof of (3.6).

Thus, inequality (3.4), together with (3.5) and (3.6) shows that

$$\alpha(\hat{P}(\hat{D})) \leq M(\tau - \sigma)[K(\sigma)H \sup_{0 \leq t \leq \sigma} \| T(t) \| + M(\sigma)]\alpha(\hat{D}),$$

for all $0 < \sigma \leq \tau$.

Finally, condition (3.1) implies that $\hat{P}$ is a condensing map.

By the fixed point theorem of Sadovski [8] we conclude that the set $\{\hat{\varphi} \in \hat{E} : \hat{P}\hat{\varphi} = \hat{\varphi}\}$ is non-empty and compact in $\hat{\mathscr{B}}$. Let φ be an element in E such that $\hat{P}\hat{\varphi} = \hat{\varphi}$. Then $\widehat{P^k(\varphi)} = \hat{\varphi}$ for every $k \in \boldsymbol{N}$ and, since

$$\| P^k \varphi - P^m \varphi \|_{\mathscr{B}} = \| \widehat{P^k \varphi} - \widehat{P^m \varphi} \|_{\hat{\mathscr{B}}} = 0,$$

$(P^k \varphi)_k$ is a Cauchy sequence in $\mathscr{B}$. Hence we infer that there exists $\psi \in E$ such that $P^k \varphi \to \psi$, as $k \to \infty$. Therefore, $P\psi = \psi$. Since P is a condensing map, we may assert that $\{\varphi \in E; P\varphi = \varphi\}$ is non-empty and compact.

Corollary 3 *Let* $\mathscr{B} := C \times L^1(g)$ *and let* $F: [0, \infty) \times \mathscr{B} \to X$ *be a continuous and τ-periodic function which satisfies a local Lipschitz condition. Assume that T is a compact semigroup and that conditions* (2.7), *a*), *b*) *and c*) *of Proposition* 5 *hold. Then there exists a $m\tau$-periodic solution of equation* (1.1).

Proof. Since F takes bounded sets into bounded sets and T is compact, then F satisfies (F_1). From Corollary 2 we obtain that the solution of (1.1)–(1.2) is defined on $[0, \infty)$. Proposition 5 implies that there exist $R > 0$ and $m \in \boldsymbol{N}$ so that the set $E := B_R[0]$ is invariant under the map $P\varphi = x_{m\tau}(\cdot, \varphi)$ and the set $\{x_t(\cdot, \varphi) : 0 \leq t \leq m\tau, \varphi \in E\}$ is bounded in $\mathscr{B}$. Furthermore, since $\mathscr{B} = C \times L^1(g)$, one can choose $m \in \boldsymbol{N}$ large enough so that the condition (3.1) with $m\tau$ in place of τ holds. Now, Theorem 2 asserts that there exists a $m\tau$-periodic solution of (1.1)–(1.2) with $\varphi \in E$.

Next, we will apply Corollary 3 to linear equations.

Example. We consider the equation

$$\dot{x}(t) = Ax(t) + F(t, x_t) + f(t), \tag{3.8}$$

where f is continuous and τ-periodic; $F(t, \cdot)$ is linear for each $t \geq 0$ and, all the assumptions considered in Corollary 3 hold. Then there exists a τ-periodic solution of (3.8). In fact, we know that there exists a $m\tau$-periodic solution $x(\cdot)$, for some $m \in N$. Let us introduce the set $V := \overline{c(\{\hat{x}_t(\cdot) : 0 \leq t \leq m\tau\})}$. Since $\{x_t(\cdot) : 0 \leq t \leq m\tau\}$ is compact in $\mathscr{B}$, then V is also a compact and convex subset of the Banach space $\hat{\mathscr{B}}$. On the other hand, if P denotes the map $P\varphi := x_\tau(\cdot, \varphi)$, then P is a continuous and affine map. The last property is a consequence of the uniqueness of solutions of equation (3.8) with initial condition (1.2). Consequently, the map $\hat{P}$ induced by P on $\hat{\mathscr{B}}$ also verifies these properties, which implies that $\hat{P}(V) \subseteq V$. Now, the Schauder's fixed point Theorem assert that $\hat{P}$ has a fixed point and, proceeding as in the roof of Theorem 2, we infer that P has a fixed point, which in turn implies that equation (3.8) has a τ-periodic solution.

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