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Asymptotic Behavior and Oscillation in Neutral Delay Difference Equations*

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1. Introduction and preliminaries

Recently there has been a lot of activity concerning the oscillation and asymptotic behavior of delay difference equations. See, for example, [2–13] and [15–16]. For the general theory of difference equations the reader is refered to [14]. We should note that the oscillatory behavior of ordinary differential equations and its discrete analogue can be quite different, for example, it is well known that every solution of the Logistic equation

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{k}\right)$$

is monotonic. But its discrete analogue

$$x_{n+1} = ax_n(1 - x_n)$$

has a chaotic solution when $a = 4$ (see [14]). In addition, the difference on the oscillation of delay differential equations and its discrete analogues also exists, see for example [16].

In this paper we consider the neutral delay difference equation

(1) $$\Delta(y_n + py_{n-k}) + q_n y_{n-1} = 0, \quad n = 0, 1, 2, \ldots$$

whose oscillation and asymptotic behavior have been investigated in [3, 4, 10–12], where p, $q_n (n = 0, 1, 2, \ldots)$ are real numbers, k and l are non-negative integers, and Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$. Eq. (1) was first considered by Brayton and Willoughby [1] from the numerical analysis point of view.

The following result has been established by Georgiou, Grove and Ladas [4].

Theorem A [4]. *Suppose*

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(2) $k \in \{1, 2, \ldots\}$, $l \in \{0, 1, 2, \ldots\}$ *and* $q_n \geqslant 0$ *eventually.*

Suppose also that

(3) $$\sum_{n=0}^{\infty} q_n = \infty.$$

Let $\{y_n\}$ *be a nonoscillatory solution of Eq.* (1). *Then the following statements are true*:

(a) *if* $p < -1$, *then* $\lim_{n \to \infty} y_n = \infty$ *or* $\lim_{n \to \infty} y_n = -\infty$;

(b) *if* $p > -1$ *and* $p \neq 1$, *then* $\lim_{n \to \infty} y_n = 0$.

Remark 1. As we have seen in [4], the assumption that $p = -1$, as well as that (2) and (3) hold, implies that every solution of Eq. (1) oscillates. Thus the assumption in the above Theorem A that $p \neq -1$ is harmless. But, the case $p = 1$ has not yet been handled. Therefore, Georgiou, Grove and Ladas posted the following conjecture in [4].

Conjecture B [4]. *Assume that* (2) *and* (3) *hold. Suppose moreover that* $p = 1$. *Let* $\{y_n\}$ *be a nonoscillatory solution of Eq.* (1). *Then*

$$\lim_{n \to \infty} y_n = 0.$$

In addition, it is also valuable to consider the following problem:

Problem C. *Whether Condition* (3) *is a necessary condition for the oscillation of all solutions of Eq.* (1) *with* $p = -1$?

Our aim in this paper is to solve the above open problems. In section 2, we first give a counterexample to show that Conjecture B is not true, and then give several positive solutions to Conjecture B under some special cases, that is, we show that, under appropriate additional hypotheses on $\{q_n\}$, Conjecture B is true. In section 3, we provide a sufficient condition which does not require Condition (3) for the oscillation of all solutions of Eq. (1) with $p = -1$.

Let $m = \max\{k, l\}$. Then by a solution of Eq. (1) we mean a sequence $\{y_n\}$ of real numbers which is defined for $n \geqslant -m$ and which satisfies Eq. (1) for $n = 0, 1, 2, \ldots$. Clearly, in this case if we given real numbers

(4) $$y_n = A_n, \; n = -m, \; -m+1, \ldots, 0$$

as a set of initial conditions, then Eq. (1) has a unique solution satisfying (4) (if it is not the case that $p = -1$ and $k = 0$).

A solution $\{y_n\}$ of Eq. (1) is said to be nonoscillatory if the terms y_n are either eventually positive or eventually negative. Otherwise, the solution is called oscillatory.

In the sequel, when we write a sequential inequality we assume that it holds for all sufficiently large positive integers.

2. Asymptotic behavior of nonoscillatory solutions

In this section we will first discuss a special example which shows that Conjecture B mentioned in Section 1 is not true, and then we prove, under appropriate additional hypotheses on $\{q_n\}$, that Conjecture B is true.

In the following, we set

$$B_n = \begin{cases} 0, & \text{if } n \text{ is an even integer,} \\ 1, & \text{if } n \text{ is an odd integer,} \end{cases}$$

Clearly, we have $B_n + B_{n-1} = 1$ for all integers n.

Now let us consider the following neutral delay difference equation

$$\Delta(y_n + y_{n-1}) + q_n y_{n-1} = 0, \qquad n = 0, 1, 2, \dots \tag{5}$$

where

$$q_n = (e^2 - 1)/(e^{n+1} B_{n-1} + e^2).$$

Obviously, $\{q_n\}$ is a sequence of positive real numbers and

$$\sum_{n=0}^{\infty} q_n = \infty.$$

Therefore, Conditions (2) and (3) are satisfied. However, we find by direct substitution that $y_n = B_n + e^{-n}$ is a positive solution of Eq. (5) and y_n does not go to zero as n goes to infinite since

$$\limsup_{n \to \infty} y_n = 1.$$

We should remark that the above example indeed shows that Conjecture B is not true. Therefore, the answer to Conjecture B mentioned in Section 1 is negative. However, the following two theorems show that, under appropriate additional hypotheses on $\{q_n\}$, Conjecture B is true.

Theorem 1. *Assume that* (2) *and* (3) *hold and that*

$$q_n > 0 \quad \textit{and} \quad \limsup_{n \to \infty} \frac{q_n}{q_{n-k}} = \beta < \infty. \tag{6}$$

Then every nonoscillatory solution of Eq. (1) *with* $p=1$ *goes to zero as* $n \to \infty$.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of Eq. (1). As $\{-y_n\}$ is also a solution of Eq. (1), we may (and do) assume that it is eventually positive. Thus, there exists an integer $n_1 \geqslant 1$ such that $q_n > 0$ and $y_{n-k} > 0$ for $n \geqslant n_1$. Set

$$z_n = y_n + y_{n-k}.$$

Then by (1) we see

$$\Delta z_n \leqslant 0, \quad z_n > 0 \qquad \text{for } n \geqslant n_1.$$

Moreover, from (1) we have

$$\Delta z_n = -q_n y_{n-1}, \quad \Delta z_{n-k} = -q_{n-k} y_{n-k-1}$$

and then we have

$$\Delta z_n + \frac{q_n}{q_{n-k}} \Delta z_{n-k} + q_n z_{n-1} = q_n(z_{n-1} - y_{n-1} - y_{n-k-1}) = 0. \tag{7}$$

From (6) there is an integer $n_2 \geqslant n_1$ such that

$$\frac{q_n}{q_{n-k}} < \beta + 1 \qquad \text{for } n \geqslant n_2.$$

Substituting this into (7) we get

$$\Delta z_n + (\beta+1)\Delta z_{n-k} + q_n z_{n-1} \leqslant 0, \qquad n \geqslant n_2. \tag{8}$$

Now, we want to prove that

$$\lim_{n\to\infty} z_n = 0.$$

Otherwise,

$$\lim_{n\to\infty} z_n = a > 0.$$

Summing up both sides of (8) from n_2 to $N(\geqslant n_2)$, we have

$$\begin{aligned} &z_{N+1} - z_{n_2} + (\beta+1)(z_{N-k+1} - z_{n_2-k}) \\ &\leqslant -\sum_{n=n_2}^{N} q_n z_{n-l} \leqslant -a\Big(\sum_{n=n_2}^{N} q_n\Big) \longrightarrow -\infty \quad \text{as} \quad N \longrightarrow \infty, \end{aligned}$$

which implies that $z_n \to -\infty$ as $n \to \infty$, a contradiction. Hence,

$$\lim_{n\to\infty} z_n = 0,$$

which implies that

$$\lim_{n\to\infty} y_n = 0.$$

The proof of this theorem is complete.

Theorem 2. *Assume that* (2) *holds and that*

$$\sum_{n=k}^{\infty} q_n^* = \infty \tag{3$'$}$$

where $q_n^* = \min\{q_n, q_{n-k}\}$. *Then every nonoscillatory solution of Eq.* (1) *with* $p = 1$ *tends to zero as* $n \to \infty$.

Proof. Let $\{y_n\}$ be an eventually positive solution of Eq. (1). Set $z_n = y_n + y_{n-k}$. Then

$$\Delta z_n = -q_n y_{n-l} \leqslant 0$$

which means that $\{z_n\}$ is eventually positive and decreasing and so the limit $\lim_{n\to\infty} z_n = M \geqslant 0$ exists and is finite. For

$$\begin{aligned} 0 &= \Delta z_n + q_n y_{n-l} + \Delta z_{n-k} + q_{n-k} y_{n-k-l} \\ &\geqslant \Delta z_n + \Delta z_{n-k} + q_n^*(y_{n-l} + y_{n-k-l}) \\ &= \Delta(z_n + z_{n-k}) + q_n^* z_{n-l} \end{aligned}$$

which means that $\{z_n\}$ satisfies the difference inequality

$$\Delta(z_n + z_{n-k}) + q_n^* z_{n-l} \leqslant 0. \tag{9}$$

From this and (3)′, and using the same arguments as that in the proof of Theorem 1, we get

$$\lim_{n\to\infty} z_n = 0$$

and so $\lim_{n\to\infty} y_n = 0$ and the proof is complete.

3. Oscillation of Eq. (1) with $p = -1$

In this section we study the oscillation of Eq. (1) with $p = -1$. The following theorem is the main result.

Theorem 3. *Assume that* (2) *holds with* $p = -1$. *Suppose also*

(10) $$\sum_{n=0}^{\infty} nq_n \sum_{j=n}^{\infty} q_j = \infty.$$

Then every solution of Eq. (1) *oscillates.*

Proof. As (3) implies that all solutions of Eq. (1) oscillate, it suffices to show that all solutions of Eq. (1) oscillate in the case that

(11) $$\sum_{n=0}^{\infty} q_n < \infty.$$

Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $\{y_n\}$. Choose an integer $n^* \geqslant 1$ to be such that

(12) $$y_{n-m} > 0 \qquad \text{for} \quad n \geqslant n^*$$

where $m = \max\{k, l\}$. Set $z_n = y_n - y_{n-k}$. Then by (1) we have

(13) $$\Delta z_n = -q_n y_{n-1} \leqslant 0 \ (\not\equiv 0) \qquad \text{for} \quad n \geqslant n^*$$

which implies that z_n is nonincreasing for $n \geqslant n^*$. Therefore, z_n is eventually nonnegative or eventually negative.

First, we assume that $z_n < 0$ eventually. Since $\{z_n\}$ is nonincreasing, there exist $a > 0$ and $N \geqslant n^*$ such that

$$z_n \leqslant -a \qquad \text{for} \quad n \geqslant N.$$

Therefore,

$$y_n \leqslant -a + y_{n-k} \qquad \text{for} \quad n \geqslant N$$

and it follows that

$$y_{N+ik} < -(i+1)a + y_{N-k} \longrightarrow -\infty \quad \text{as} \quad i \longrightarrow \infty$$

which contradicts (12). Thus, z_n can not be eventually negative.

Next, we assume that $z_n \geqslant 0$ eventually. In this case we have eventually $y_n \geqslant y_{n-k}$, which means that there exist $M > 0$ and $N^* \geqslant N$ such that $y_{n-m} \geqslant M$ for $n \geqslant N^*$. Then from (13), it follows that

$$\Delta z_n \leqslant -Mq_n \qquad \text{for} \quad n \geqslant N^*.$$

Hence

$$z_n \geqslant M \sum_{j=n}^{\infty} q_j \qquad \text{for} \quad n \geqslant N^*$$

and so

$$y_n \geqslant y_{n-k} + M \sum_{j=n}^{\infty} q_j \qquad \text{for } n \geqslant N^*. \tag{14}$$

Now let $I(n)$ denote the greatest integer part of $(n - N^*)/k$, then we have

$$y_n \geqslant M(\sum_{j=n}^{\infty} q_j + \sum_{j=n-k}^{\infty} q_j + \cdots + \sum_{j=n-(I(n)-1)k}^{\infty} q_j) + y_{n-I(n)k}$$

which, together with (13), yields

$$\Delta z_n \leqslant -I(n) M q_n \sum_{j=n}^{\infty} q_j := -H_n. \tag{15}$$

By noting the fact that $I(n)/n \to 1/k$ as $n \to \infty$, we see that

$$H_n(nq_n \sum_{j=n}^{\infty} q_j)^{-1} = \frac{I(n)M}{n} \longrightarrow \frac{M}{k} \quad \text{as} \quad n \longrightarrow \infty. \tag{16}$$

Clearly, (10) and (16) imply that

$$\sum_{n=0}^{\infty} H_n = \infty$$

which, together with (15), yields

$$z_n \longrightarrow -\infty \quad \text{as} \quad n \longrightarrow \infty$$

which contradicts the hypothesis that z_n is eventually nonnegative. Thus, all solutions of Eq. (1) oscillate. The proof is complete.

Remark 2. Clearly, (10) is weaker than (3). Hence, Theorem 3 is an improvement of Theorem 1 in [4] and shows that the answer to Problem C mentioned in Section 1 is negative.

Example. Consider the neutral difference equation

$$\Delta(y_n - y_{n-k}) + n^{-a} y_{n-l} = 0 \tag{17}$$

where $a \in (1, 3/2]$. Since

$$n^{-a}(n^{-(a-1)} - (n+1)^{-(a-1)})^{-1} \longrightarrow \frac{1}{a-1} \quad \text{as} \quad n \longrightarrow \infty$$

and $n^{-(a-1)} - (n+1)^{-(a-1)}$ satisfies condition (10), it is easy to see that n^{-a} satisfies (10). Therefore, by Theorem 3, every solution of Eq. (17) oscillates. But, condition (3) does not satisfy.

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