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The Stokes' Multipliers and the Galois' Group of a Non-Fuchsian System and the Generalised Phragmen-Lindelöff Principle

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1. Introduction

In this paper we consider non-fuchsian linear systems, i.e. systems of ordinary linear differential equations of the kind

$$t^r X' = A(t)X \tag{1}$$

where $r \geq 2$ is an integer, $A \in gl(n, \boldsymbol{C}\{t\})$, $A(0) \neq 0$, "$'$" $\equiv d/dt$ (by definition, system (1) is called *non-fuchsian*, because it has a pole of order ≥ 2 at 0). In the next section we make a brief review of some well-known facts about the asymptotics and the analytic properties of the solutions of system (1) at 0 and define the Stokes' multipliers and the monodromy operator which are linear operators acting in the solution space of system (1).

Denote by M the field of holomorphic or meromorphic functions in the neighbourhood of 0, with a single pole at 0. Denote by $S \subset \boldsymbol{C}$ a sector with vertex at 0 and by K_S the extension of M obtained by adjoining all the restrictions to S of the components of the solutions of system (1). The *Galois' group* of system (1) is the group of automorphisms of the differential field K_S which preserve the field M. One of the aims of this paper is to prove

Theorem 1.1. *The Stokes' multipliers and the monodromy operator of system* (1) *belong to its Galois' group.*

Remark: In fact, we prove Theorem 1.1 not for system (1), but for system (3) obtained from it by some holomorphic or fractionally meromorphic in t linear transformation of the dependent variables X as described in Section 2.

This theorem was stated by prof. J.-P. Ramis, see [1], [2], who also gave a sketch of the proof there. The proof is going to be published in a book. It

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is using the so-called resummation method which has also been applied in the sketch of the proof of the finiteness of the limit cycles theorem announced by prof. J. Ecalle, prof. J. Martinet, prof. R. Moussu and prof. J.-P. Ramis, see [3]. An independent proof (for the non-resonant case) was given by prof. Yu. S. Il'yashenko and prof. A. G. Hovansky, see [4]; it is based on the theory of the functional cochains developed by the first author. We list some papers of prof. J. P. Ramis, ses Additional References, from which one can derive the proof of Theorem 1.1 and many other results concerning non-fuchsian equations and their differential Galois theory.

Naturally, the author of this paper does not claim any priority in proving Theorem 1.1. His aim is to give an independent proof including the resonant case and to show some results about the dependence of the angular type of holomorphic functions of a given order in sectors on the angle. 'Independent' is relative, of course—the problem was stated to the author by prof. Yu. S. Il'yashenko and the author is following the same scheme of reasoning in Section 3, C) as the one used in [4].

In Section 2 we expose briefly some well-known facts from the local analytic theory of non-fuchsian linear systems. Section 3 is devoted to the Stokes' multipliers, to the differential field K_S and to the Galois' group of system (3) which is in some sense equivalent to system (1). In Section 4 we describe the behaviour of the angular type of holomorphic functions in sectors of a given order and formulate the generalised Phragmen-Lindelöff principle; the proofs of two of the theorems are exposed in the Appendix. In Section 5 we prove Theorem 3.2 which is equivalent to Theorem 1.1. We don't make use of all the results in Section 4 in the proof of Theorem 3.2, but they are closedly related to the problem of describing the behaviour of the solutions of a non-fuchsian system.

2. Local normal forms and analytic theory of non-fuchsian linear systems (brief review)

The local analytic theory of non-fuchsian linear systems is well exposed in [5]. For the local normal forms we refer the reader to [6] and [7]. Non-fuchsian systems from a more algebraic point of view are considered in [8] and [9].

1) System (1) is called *resonant* if there are equal among the eigenvalues of the matrix $A(0)$. For any non-resonant (resonant) system (1) there exists a formal change of variables

$$X \mapsto Q(t)X \qquad (X \mapsto Q(t^{1/m})X, m \in N)$$

where the matrix Q is a formal Taylor power series of the variable t, $\det Q(0) \neq 0$ (a formal Laurent power series of the variable $t^{1/m}$) which transforms system (1) to the form

$$(2) \qquad \tau^q X' = (D_0 + D_1\tau + \cdots + D_{q-2}\tau^{q-2} + C\tau^{q-1})X$$

Here "$'$" $\equiv d/d\tau$, $\tau = t$ in the non-resonant and $\tau = t^{1/m}$ in the resonant case, the matrices $D_0, \ldots, D_{q-2}$ are diagonal and the matrix C is in Jordan normal form, the invariant spaces of the linear operators with matrices $D_0, \ldots, D_{q-2}$, C are embedded in each other according to the rule

$$\mathrm{Inv}\, D_0 \supset \mathrm{Inv}\, D_1 \supset \cdots \supset \mathrm{Inv}\, D_{q-2} \supset \mathrm{In}\, C$$

Hence, the matrices $D_0, \ldots, D_{q-2}, C$ commute. In the non-resonant case the diagonal entries of the matrix D_0 are exactly the eigenvalues of the matrix $A(0)$.

Any solution of the formal normal form (3) is of the kind

$$X = \exp(\mathrm{diag}\,(b_1(1/\tau), \ldots, b_n(1/\tau)) \exp(C \ln \tau)G\,,$$

where $G \in GL(n, \boldsymbol{C})$ and $b_1, \ldots, b_n$ are polynomials of $1/\tau$ of power $q-1$ without constant terms.

In the non-resonant case $\det Q(0) \neq 0$, $q = r$. In the resonant one, in general, $q > r$. In both cases the matrices $\mathrm{diag}\,(b_1, \ldots, b_n)$ and C commute.

2) In the normalizing transformation we can cut off the sufficiently large powers of τ and obtain a holomorphic (or fractionally meromorphic—in the resonant case) transformation which transforms system (1) to the form

$$(3) \qquad \tau^q X' = (D_0 + D_1\tau + \cdots + D_{q-2}\tau^{q-2} + C\tau^{q-1} + \tau^q W(\tau))X$$

where $W \in gl(n, \boldsymbol{C}\{\tau\})$.

System (3) is non-ramified and further we consider this system instead of system (1). In [5] one can find an analytic presentation of the solution of system (1) in the resonance ramified case.

3) The *Stokes' lines* of system (3) in the non-resonant case are defined as follows: put

$$b_j(1/\tau) = b_{j,q-1}/\tau^{q-1} + b_{j-2}/\tau^{q-2} + \cdots + b_{j,1}/\tau$$

Then the Stokes' lines are defined by

$$\mathrm{Re}\,[(b_{j,q-1} - b_{k,q-1})/\tau^{q-1}] = 0\,, \qquad j \neq k$$

To each pair of polynomials $(b_j(1/\tau), b_k(1/\tau))$ there correspond exactly $2(q-1)$ such lines (in fact, what we call Stokes' lines are rays beginning at 0). In the resonant case the Stokes' lines are defined by

$$\mathrm{Re}\,[(b_{j,s} - b_{k,s})/\tau^s] = 0$$

where $b_{j,l} = b_{k,l}$ for $l > s$ and $b_{j,s} \neq b_{k,s}$. Therefore in the resonant case to the pair $(b_j(1/\tau), b_k(1/\tau))$ there correspond exactly $2s$ Stokes' lines where $s \leq q - 1$. The number s is called the *level* of the pair (b_j, b_k) and of the Stokes' lines corresponding to it.

4) Consider a sufficiently narrow sector S with vertex at 0; 'sufficiently narrow' means not containing two Stokes' lines corresponding to one and the same pair (b_j, b_k). Then there exists a holomorphic matrix-function $H(\tau)$: $S \to S$ having an asymptotic expansion at 0 as a formal Taylor series (or Laurent series—in the resonant case) coinciding with the one of the normalizing transformation, see 1), such that the general solution to system (3) admits the presentation (for $\tau \in S$)

$$X = H(\tau) \exp(\mathrm{diag}\,(b_1(1/\tau), \dots, b_n(1/\tau)) \exp(C \ln \tau) G \tag{4}$$

with $G \in GL(n, \boldsymbol{C})$. Further, for the sake of simplicity, we consider only the case $G = I \equiv \mathrm{diag}\,(1, \dots, 1)$.

Remark: Further in the text we refer to (4) both as to an analytic presentation of the solution and as to its asymptotics at 0; it should be clear from the context which of the two is meant.

If the sector S does not contain any Stokes' line corresponding to some of the pairs (b_j, b_k), then the presentation (4) does not define a unique solution to system (3). Really, suppose that system (3) is not resonant (in the resonant case the reasoning is similar). Suppose that the sector S contains no Stokes' line corresponding to the pair (b_1, b_2). Let the solution $\tilde{X}$ to system (3) have the expansion (4) in S. Then the solution $\tilde{X}V$ has the same expansion where

$$V = \begin{bmatrix} 1 & a & \dots & 0 \\ b & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Here $a = 0$ and $b \in \boldsymbol{C}$ is arbitrary, if $\mathrm{Re}\,(b_1 - b_2) > 0$ for $\tau \in S$ sufficiently small and $b = 0$, $a \in \boldsymbol{C}$ is arbitrary in the opposite case. It is easy to see that the multiplication by V to the right can be replaced by changing $H(\tau)$; the new matrix-function $H(\tau)$ differs by exponentially small for $\tau \to 0$ terms and hence, has the same asymptotics at 0.

In the non-resonant case the asymptotics (4) defines a unique solution to system (3) if and only if the sector S contains for each pair (b_j, b_k) exactly one Stokes' line corresponding to it, strictly inside itself. Such sectors are called *nice*. In the resonant case we call nice a maximal sector that for each of the pairs (b_j, b_k) contains no more than one Stokes' line corresponding to it. It is possible that some of the nice sectors contain Stokes' lines corresponding not to every such pair. In such sectors the asymptotic expansion (4) defines infinitely many solutions the matrices H of which differ by exponentially small for $\tau \to 0$ terms.

3. The differential field K_S, the Stokes' multipliers and the Galois' group of system (3)

A) We define the *Stokes' multipliers* here. Consider the non-resonant case first. Consider for any two overlapping nice sectors the solutions with equal asymptotics (4) in them. Then these solutions are obtained from each other by multiplying from the right by some constant non-degenerate matrix.

Let the solutions be W_1, W_2, defined on the sectors S_1, S_2. Then the matrix V has units on the diagonal. It may have non-zero elements exactly on those off-diagonal positions (j, k) for which $\operatorname{Re}(b_k(1/\tau) - b_j(1/\tau)) < 0$ for $\tau \in S_1 \cap S_2$ and τ sufficiently small. Note that one of the sides of S_1 (contained in S_2) is a Stokes' line corresponding to the pair (b_j, b_k). This pair is said to *change dominance* when τ intersects the Stokes' line. The matrix V is called a *Stokes' multiplier*. Given the sectors S_1, S_2, it is defined in a unique way. The matrices H of the solutions W_1, W_2, see (4), differ in $S_1 \cap S_2$ by terms which are $O(\exp(-c/|\tau|^{q-1})$, $c > 0$ for $\tau \to 0$.

Consider the resonant case. In this case the solutions W_1, W_2 may not be uniquely defined by the asymptotics (4) and, hence, the Stokes' multiplier V, too, see part 4) of Section 2. It may again contain non-zero entries exactly on those off-diagonal positions (j, k), for which $\operatorname{Re}(b_k(1/\tau) - b_j(1/\tau)) < 0$ for $\tau \in S_1 \cap S_2$ and $|\tau|$ sufficiently small.

Fix the solution W_1. We say that the Stokes' multiplier V has *minimal support* if it does not contain non-zero off-diagonal entries on those positions (j, k), for which the pair (b_j, b_k) does not change dominance in S_2. This means that if V has not minimal support, then the matrices H of the solutions W_1, W_2, see (4), differ in $S_1 \cap S_2$ by terms which are $O(\exp(-1/|\tau|^l))$, $l < q - 1$. If V has minimal support, then the number l is the minimal of the levels of the pairs (b_j, b_k) corresponding to Stokes' lines which are contained in $S_2 \backslash S_1$.

B) Theorem 3.1. *Let S_1, S_2, ... be a sequence of nice overlapping sectors on the universal covering of $\boldsymbol{C} \backslash 0$, their order corresponding to τ encircling* 0 *anticlockwise. Let W_1', W_1'' be two different matrix solutions to system* (3) *with the same asymptotics* (4) *in S_1. Denote by W_2', W_2'' the solutions to system* (3) *with this asymptotics in S_2 obtained from W_1', W_1'' by multiplying with the Stokes' multipliers with minimal support. Construct the pairs (W_3', W_3''), (W_4', W_4''), ... in the same way. Then for m sufficiently large we have $W_m' \equiv W_m''$.*

This theorem is proved in [5]. Here we give a sketch of the proof. W_1' and W_1'' differ in S_1 by terms which are $O(e^{-c/|\tau|^l})$. When we pass from S_1 to S_2, i.e. from (W_1', W_1'') to (W_2', W_2''), then the Stokes' multipliers remove those terms in (W_1', W_1'') which correspond to pairs (b_j, b_k) having change of dominance in S_2 (and they don't introduce other exponentially small in S_2 differences between W_2' and W_2''—this is the sense of the minimal support). After sufficiently many steps W_m' and W_m'' do not differ at all.

C) We expose this part in the same way as it is exposed in [4], omitting the proofs. Suppose that a nice sector S is fixed (the choice of another sector changes the Galois' group of system (3) to a conjugate one, the conjugation being done by the operator of analytic continuation). The Galois' group admits an exact n-dimensional representation. Its automorphisms induce a transformation of the Cartesian power $K_S^n \to K_S^n$; this representation maps the solution space of system (3) onto itself. We remind that we prefer to be dealing with the non-ramified situation, i.e. with system (3) instead of system (1).

Let A be an arbitrary and B—a matrix with non-negative integer elements. Put

$$A^B = \prod_{i,j} a_{ij}^{b_{ij}}$$

(this is an analog of the usual multiindices). Then every element of the field K_S has the form

$$f = \sum a_K z^K \Big/ \sum b_K z^K \tag{5}$$

where the matrix K has non-negative integer elements with values in some dependent on f finite set. Equation (3) allows us to replace differentiating by algebraic operations.

Consider a basis Z of the solution space $\mathscr{L}$ of system (3) (more precisely, of its restriction to the sector S). Then there corresponds a linear operator T_L to the automorphism $L - T_L: Z \mapsto ZT_L$ (we denote its matrix also by T_L). We have

$$Lf = \sum (ZT_L)^K a_K \Big/ \sum (ZT_L)^K b_K \tag{6}$$

This formula, however, does not allow to construct an automorphism after any operator $T_L: \mathscr{L} \to \mathscr{L}$. If the representation (5) is not unique, then the element (6) will not be defined correctly. For the element (6) to be defined correctly it is necessary and sufficient that the operator L should preserve the relations in the differetial field K_S, i.e. if the equality

$$\sum a_K Z^K = 0 \tag{7}$$

holds, then there must hold the equality

$$\sum (ZT_L)^K a_K = 0$$

as well. In other words, the operator L preserves any algebraic equality involving the components of the solutions to system (3).

Theorem 1.1 can be equivalently reformulated as follows:

Theorem 3.2. *The Stokes' multipliers and the monodromy operator preserve the relations in the differential field* K_S.

For the monodromy operator the theorem is trivial—the analytic continuation of the solutions preserves the relations. We prove Theorem 3.2 in Section 5, after developing some necessary apparatus in Section 4.

Remark: Note that the changing of the sector S changes K_S to an isomorphic differential field. Therefore we do not distinguish the fields K_S corresponding to the different sectors S.

4. On the growth rate of holomorphic functions in sectors

A) In this section we consider functions defined on sectors in $\boldsymbol{C}$ of the kind $S = \{z \in \boldsymbol{C} \mid a \le \arg z \le b\}$, holomorphic inside and continuous on their closure, and their growth rate for $|z| \to \infty$. All the results are directly transferred to the case when $|z| \to 0$ by the change of variables $z \mapsto 1/z$.

The well-known Phragmen-Lindelöff principle states that if a holomorphic function is decreasing as $\exp(-c|z|^\alpha)$, $c > 0$ for $|z| \to \infty$ uniformly in such a sector, then its opening must be less than π/α (for the precise formulation see [10]). In this section we consider the growth rate of holomorphic functons in sectors and its changing with the angle. We are going to examine the behaviour of the angular type of functions of a given order. We don't use the notion of an order but the following definition:

A function f is said to belong to the class B_α, if it is defined on a sector of opening $\le \pi/\alpha$ and there exist constants $c > 0$, $d > 0$ such that the inequality $|f| < ce^{d|z|^\alpha}$ holds. Further we denote the sector by $S = \{z \mid a \le \arg z \le b, b - a < \pi/\alpha\}$. For $f \in B_\alpha$ put

$$\rho_f(\omega) = \inf\{k \in \boldsymbol{R} \mid \exists c(k) > 0 \colon |f(re^{i\omega})| < c(k)e^{kr^\alpha}, r \ge 0, a \le \omega \le b\}\,.$$

We call this number the *angular type of f corresponding to the direction ω.* Put

$$A_\alpha^+ = \{f \in B_\alpha | \rho_f(\omega) \geq 0 \ \forall \omega \in [a, b]\}$$

$$A_\alpha^- = \{f \in B_\alpha | \rho_f(\omega) \leq 0 \ \forall \omega \in [a, b]\}$$

$$\tilde{A}_\alpha^- = \{f \in B_\alpha | \rho_f(\omega) < 0 \ \forall \omega \in [a, b]\}$$

$$A_\alpha = \{f \in B_\alpha | -\infty < \rho_f(\omega) < \infty \ \forall \omega \in [a, b]\}$$

Example: For $g = e^{sz^\alpha}$ we have

$$\rho_g(\omega) = |s| \cos(\alpha \arg z + \arg s)$$

Note that $g \in B_\alpha$ on any sector with vertex at 0 of its Riemann surface.

The following evident lemma is true:

Lemma. 1) *If $f, g \in B_\alpha$ and if $\rho_f(\omega) > \rho_g(\omega)$, then $\rho_{f+g}(\omega) = \rho_f(\omega)$.*
2) *If $f \in B_\alpha$, $g = e^{cz^\alpha}$, $c \in \boldsymbol{C}$, then $\rho_{fg}(\omega) = \rho_f(\omega) + \rho_g(\omega)$*
3) *For any $d \in \boldsymbol{C} \backslash 0$ we have $\rho_f(\omega) = \rho_{df}(\omega)$.*

Example: $\rho_{\sin z}(\omega) = |\sin \omega| = \max(\sin \omega, -\sin \omega) = \max(\rho_g(\omega), \rho_h(\omega))$, $g = e^{iz}$, $h = e^{-iz}$.

General remark: Most of the proofs in this paper are based on the comparison of a function $\rho_f(\omega)$ with the function $\rho_g(\omega)$ for $g = e^{cz^\alpha}$, making use of the lemma.

Theorem 4.1. *There exists no function $f \in B_\alpha$ defined for $\{a \leq \arg z \leq b,$ $b - a < \pi/\alpha\}$ such that the following inequalities hold simultaneously*:

$$\rho_f(a) < \rho_f(\omega) < 0\,, \qquad \rho_f(b) < \rho_f(\omega) < 0$$

for some $\omega \in (a, b)$.

Remark: Theorem 4.1 stays correct if we allow the equality $\rho_f(a) = -\infty$ or $\rho_f(b) = -\infty$ or both (with $\rho_f(\omega) > -\infty$).

Proof: 1° Without loss of generality we put $\alpha = 2$ (this can always be achieved by the change of variables $z \mapsto z^{2/\alpha}$). Suppose that such a function f exists. Then for $\arg z = a$ and $\arg z = b$ the estimation

$$|f| \leq Ce^{-q|z|^2}\,, \qquad q > 0,\ C > 0\,,$$

holds and on the ray $\arg z = \omega$ the estimation

$$e^{-p|z|^2} < |f| < De^{-p'|z|^2}$$

holds, where $D > 0$, $0 < p' \leq p < q$; the right inequality holds for all z with $\arg z = \omega$ and the left is satisfied by infinitely many z with $\arg z = \omega$ and $|z|$ arbitrarily large.

Denote the rays $\arg z = a$, $\arg z = \omega$, $\arg z = b$ by l_0, l_1, l_2 respectively; the angles (l_0, l_1), (l_1, l_2)—by ϕ_1, ϕ_2. Without loss of generality we put $\omega = 0$, i.e. the ray l_1 is the positive real semi-axis, and we assume that $\phi_1 < \pi/4$, $\phi_2 < \pi/4$, see 5°.

2° Consider the function

$$g = e^{-\sigma^2/4(s+p)} e^{-sz^2+\sigma z}$$

We make an estimation for $|fg|$ and (making use of the decreasing of f and g at ∞) show that $|fg|$ is maximal strictly inside S which contradicts the maximum principle.

3° Here we define the functions h_0, h_1 which estimate the growth rate of $|fg|$ for $z \in l_0$, l_1, l_2, see 4°. For the function

$$h_0(x) = e^{-\sigma^2/4(s+p)} e^{-(p+s)x^2+\sigma x}$$

we have

$$\max_{x \geq 0} h_0 = h_0(\sigma/2(s+p)) = 1$$

Consider the function

$$h_1(x) = Ce^{-q(1+k^2)x^2} e^{-\sigma^2/4(s+p)} e^{-s(1-k^2)x^2+\sigma x}$$

where k with $|k| < 1$ is a real parameter. Then

$$\max_{x \geq 0} h_1 = h_1(\sigma/2(q(1+k^2)+s(1-k^2))) = Ce^{[-\sigma^2/4(s+p)]+[\sigma^2/4(q(1+k^2)+s(1-k^2))]}$$

For σ sufficiently large ($\sigma > 0$) and $p < s < q$ this value is smaller than 1, because

$$-\frac{1}{4(s+p)} + \frac{1}{4(q(1+k^2)+s(1-k^2))} < 0$$

(this follows from $-q(1+k^2) - s(1-k^2) + s + p = -(q-p) - (q-s)k^2 < 0$).

4° The functions h_0, h_1 provide an estimation for the values of $|fg|$. Really, for infinitely many $z \in l_1$ with $|z|$ arbitrarily large we have

$$|fg| > e^{-(p+s)|z|^2+\sigma|z|} e^{-\sigma^2/4(s+p)}$$

and for $z \in l_0$, l_2 we have

$$|fg| \leq Ce^{-\sigma^2/4(s+p)} e^{-q(1+k^2)x^2} e^{-s(1-k^2)x^2+\sigma x}$$

where $k = \tan \phi_1$ or $k = \tan \phi_2$, $x = \operatorname{Re} z$, $kx = \operatorname{Im} z$, $1 + k^2 = |z|^2$.

Fix $s \in (p, q)$. Fix $\sigma_0 > 0$ such that for $\sigma > \sigma_0$ the inequality $h_0(\sigma/2(s+p)) = 1 > h_1(x)$ holds for any $x \geq 0$. Choose $\sigma > \sigma_0$ so that $|f(\sigma/2(s+p))| > e^{-p\sigma^2/4(s+p)^2}$. Then

$$|fg|(\sigma/2(s+p)) > h_0(\sigma/2(s+p)) = 1 > \max_{x \geq 0} h_1 \geq \max_{z \in l_0, l_2} |fg|$$

Consider a circumference $\tilde{c}$ of radius $r \gg \sigma/2(s+p)$ centered at 0 on which $|fg|$ is smaller than 1 (such a circumference exists, due to the rate of decreasing of $|fg|$ at ∞). Then the holomorphic function fg has a maximal module for $z = \sigma/2(s+p)$ in the domain bounded by l_0, l_2 and $\tilde{c}$ which contradicts the maximum principle.

5° It remains to prove that we can assume $\phi_1 < \pi/4$ and $\phi_2 < \pi/4$. Suppose that l_1 is not the bisector of the angle (l_1, l_2) (if it is, then there is nothing to prove). Denote the bisector by l_3 and by ω_0 the angle (l_1, l_3), $\omega_0 = (a+b)/2$. Put $-v = \rho_f(\omega_0)$. If $v \leq p$, then we can consider the rays (l_0, l_3, l_2) instead of the rays (l_0, l_1, l_2). Let $v > p$ and let l_3 be internal for the angle (l_0, l_1). Then we can consider the triple (l_3, l_1, l_2) instead of (l_0, l_1, l_2). The angles (l_3, l_1) and (l_1, l_2) are less than $\pi/4$. This completes the proof of the theorem.

Corollary 4.1. *There exists no function $f \in B_\alpha$ such that*

$$\rho_f(a') < 0\,, \qquad \rho_f(b') > 0\,, \qquad \rho_f(c') < 0$$

where $a < a' < b' < c' < b$, $b - a < \pi/\alpha$.

The corollary follows from Theorem 4.1 if we multiply f by e^{cz^α} for a suitably chosen $c \neq 0$.

Corollary 4.2. *There exists no function $f \in B_\alpha$ such that $\rho_f(\omega_0) = -\infty$, $\rho_f(\omega_1) > -\infty$; ω_0, $\omega_1 \in (a, b)$.*

Really, suppose that $\omega_0 < \omega_1$. If for some $\omega_2 > \omega_1$ we have $\rho_f(\omega_2) = -\infty$, then this would contradict Theorem 4.1, see the remarks after it. If $\rho_f(\omega_2) > -\infty$, then there exists a function $g = e^{sz^\alpha}$, $s \in \boldsymbol{C}\backslash 0$ such that $\rho_f(\omega)$ and $\rho_g(\omega)$ are such as shown on Fig. 1. It is easy to see that $\rho_{fg^{-1}}(\omega)$ provides a contradiction with Theorem 4.1.

B) Theorem 4.2. *Let $f \in A_\alpha$. Then the number $\rho_f(\omega)$ is a continuous function of the angle ω.*

Proof: 1° Continuity is a local property. Therefore we can consider holomorphic functions defined on arbitrarily narrow sectors. Without loss of

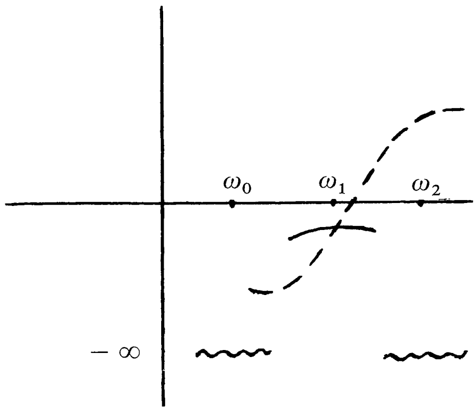

Fig. 1

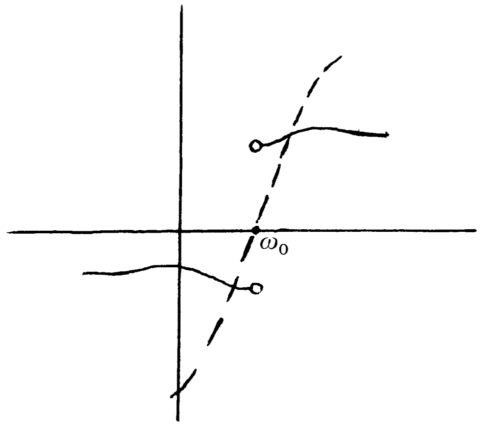

Fig. 2A

generality we prove the theorem for $f \in \tilde{A}_\alpha^- \cap A_\alpha$ which can be achieved by multiplying f by $e^{\gamma z^\alpha}$ for a suitable $\gamma \in \boldsymbol{C}$.

2° Suppose that $\rho_f(\omega)$ is discontinous for $\omega = \omega_0$. Note that $\rho_f(\omega)$ is bounded. Hence, there either exist two sequences $\{\omega_j^\pm\}$, $\omega_j^\pm \to \omega_0$, $\omega_j^+ > \omega_0$, $\omega_j^- < \omega_0$ such that

$$\rho_f(\omega_j^+) \to \rho^+ , \qquad \rho_f(\omega_j^-) \to \rho^- , \qquad \rho^+ \neq \rho^-$$

or there exists $\lim \rho_f(\omega)$ for $\omega \to \omega_0$ which is different from $\rho_f(\omega_0)$. Suppose that $\rho^+ < \rho^-$ (in the opposite case the proof is similar). Consider the function $g = fe^{-\rho_f(\omega_0)(zq)^\alpha - (zq)^\alpha r}$ where $|q| = 1$, $\arg (zq)^\alpha = \pi/2$ for $\arg z = \omega_0$, i.e. $\alpha\omega_0 + \alpha \arg q = \pi/2$, $r > 0$. Then $\rho_g(\omega_0) = 0$ and $\rho_g(\omega)$ changes sign at least thrice in the neighbourhood of ω_0 for r sufficiently large. The graphs of $\rho_f(\omega)$ and $\rho_g(\omega)$ are shown on Fig. 2A.

3° Suppose that for $a' < b' < c'$ we have $\rho_g(a') < 0$, $\rho_g(b') > 0$, $\rho_g(c') < 0$, $|a' - c'| < \pi/\alpha$. This contradicts Corollary 4.1. If $\rho_g(a') > 0$, $\rho_g(b') < 0$, $\rho_g(c') > 0$, then there exists $d' > c'$ or $d' < a'$ such that $\rho_g(d') < 0$ which again contradicts Corollary 4.1.

4° Suppose now that $\rho_f(\omega_0) \neq \lim_{\omega \to \omega_0} \rho_f(\omega)$. Then for a suitable function $h = e^{sz_\alpha}$ we would have $\rho_{fh}(d') < 0$, $\rho_{fh}(b') > 0$, $\rho_{fh}(c') < 0$ which leads to a contradiction with Corollary 4.1. An example of the graphs of $\rho_f(\omega)$ and $\rho_h(\omega)$ is shown on Fig. 2B.

Denote by $PC^+[a, b]$ the class of functions defined on some closed interval $[a, b]$, $b - a < \pi/\alpha$ of the kind

$$F(\omega) = \max(\rho_{g_1}(\omega), \dots, \rho_{g_k}(\omega)) , \qquad k \in \boldsymbol{N} , \ g_j = e^{k_j z^\alpha}$$

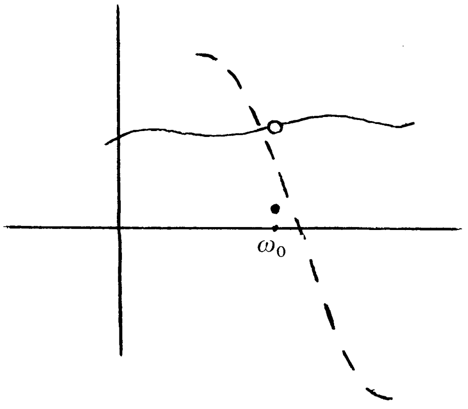

Fig. 2B

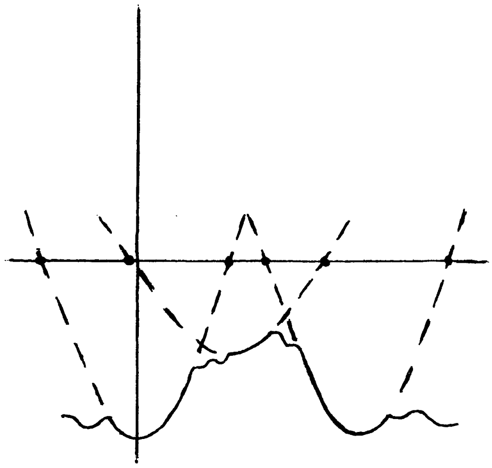

Fig. 3

A function $f\colon [a, b] \to \boldsymbol{R}$ is said to be approximated by functions of a given class $\tilde{U}$ if for any $\delta > 0$ there exists $g \in \tilde{U}$ such that

$$\|f - g\| < \delta\,, \qquad \|\cdot\| = \sup_{[a,b]} |\cdot|$$

Given a class of functions we define its closure as the set of functions which can be approximated by functions of the given class; naturally, all the functions are assumed to be defined on one and the same closed interval. Denote by $\overline{PC^+[a, b]}$ the closure of the class $PC^+[a, b]$.

Theorem 4.3. *Consider the class U of functions which are $\rho_g(\omega)$ for some function $g \in A_\alpha$ where g is defined on some fixed sector $a \le \arg z \le b$ of opening $< \pi/\alpha$. Then $U \equiv \overline{PC^+[a, b]}$.*

Theorem 4.4. *$\overline{PC^+[a, b]}$ can be equivalently defined as the class of functions f which are 'subsinusoidal', i.e. continuous on $[a, b]$ and such that if for a function of the kind $h = k \sin(\alpha\omega + d)$ we have*

$$h(\omega_1) = f(\omega_1)\,, \qquad h(\omega_2) = f(\omega_2)\,, \qquad a \le \omega_1 < \omega_2 \le b$$

then $f(\omega) \le h(\omega)$ for any $\omega \in (\omega_1, \omega_2)$.

Theorems 4.3 and 4.4 are proved in the Appendix. They are not used in the proof of Theorem 1.1.

Remark: Note that the property to be subsinusoidal, see Theorem 4.4, is similar to the property to be convex—the sinusoids play the role of the lines. Note also that the sinusoids are meant to be of one and the same period—$2\pi/\alpha$. Any two sinusoids of period $2\pi/\alpha$ intersect each other no more than once on any interval of length $< \pi/\alpha$, similarly to the lines.

Theorem 4.5. *Suppose that the holomorphic functions $f_j \in \tilde{A}_\alpha^- \cap A_\alpha$, $j = 1, \ldots, k$ are defined on the overlapping sectors S_j, $S_j \cap S_{j+1} \neq \emptyset$ with vertex at 0. Let $g_j \colon S_j \cap S_{j+1} \to \boldsymbol{C}$, $g_j \equiv f_j - f_{j+1}$, $g_j \in \tilde{A}_\beta^-$, $\beta > \alpha$, $j = 1, \ldots, k-1$. Let the opening of $S_1 \cup \cdots \cup S_k$ be $> \pi/\alpha$. Then $f_j \equiv 0$, $j = 1, \ldots, k$.*

This theorem is similar to the Phragmen-Lindelöff principle. Before proving it we shall state the Generalised Phragmen-Lindelöff Principle. It is proved at the end of Section 5. Most of its proof is contained in the proof of Proposition 4 in Section 5.

The Generalised Phragmen-Lindelöff Principle. *Let the holomorphic functions $f_j \in \tilde{A}_\alpha^-$, $f_j \notin \tilde{A}_\gamma^-$, $\gamma > \alpha$, $f_j \colon S_j \to \boldsymbol{C}$ where the sectors S_j are as in Theorem 4.5. Let $g_j \in \tilde{A}_\beta^-$, $\alpha < \gamma < \beta$, $g_j = f_j - f_{j+1}$, $g_j \colon S_j \cap S_{j+1} \to \boldsymbol{C}$. Let the opening of $S_1 \cup \cdots \cup S_k$ be $> \pi/\alpha$. Then $f_j \equiv 0$, $j = 1, \ldots, k$.*

Proof: 1° Let $S_j = \{z \mid a_j \leq \arg z \leq b_j\}$. Suppose that $f_j \not\equiv 0$. It follows from $g_j \in \tilde{A}_\beta^-$, $\beta > \alpha$ that the functions $\rho_{f_j}(\omega)$ and $\rho_{f_{j+1}}(\omega)$ coincide for $\omega \in [a_{j+1}, b_j]$, $j = 1, \ldots, k-1$. Then the function $\rho(\omega) \equiv \rho_{f_j}(\omega)$ for $\omega \in [a_j, b_j]$, $j = 1, \ldots, k$ is well defined and continuous on $[a_1, b_k]$. It takes negative values only—this follows from $f_j \in \tilde{A}_\alpha^-$.

2° Put $\delta = \min(b_1 - a_2, \ldots, b_{k-1} - a_k, a_2 - a_1, \ldots, a_k - a_{k-1}, \pi/2\alpha)$. The graph of $\rho(\omega)$ contains no more than a finite number of arcs of graphs of functions of the kind $C \cos(\alpha\omega + d)$ defined on intervals of length $> \delta/2$. The graphs of the analytic continuations of these functions intersect $[a_1, b_k]$ in a finite number of points the set of which we denote by P, see Fig. 3.

3° Put $g = e^{Cz^\alpha}$. Remember that $\rho_g(\omega) = |C| \cos(\alpha\omega + \arg C)$. We show that it is possible to choose C in such a way that there should exist an interval $[a', b'] \subset [a_1, b_k]$, $b' - a' < \delta/2$ such that $\rho(\omega_0) > \rho_g(\omega_0)$ for some $\omega_0 \in (a', b')$, $\rho(\omega) < \rho_g(\omega)$ for $\omega = a' - 0$ and for $\omega = b' + 0$. This means that the functon gf_j (where f_j is defined for $\omega \in (a', b')$; such a j exists, because $b' - a' < \delta/2$) provides a contradiction with Corollary 4.1.

4° To choose the necessary C it is necessary and sufficient to require that the number $\arg C + s\pi/2$ does not belong to P ($s \in \boldsymbol{Z}$) and then to vary $|C|$, see Fig. 4. For the large values of $|C|$ there exists an open set in $[a, b]$ on which $\rho(\omega) > \rho_g(\omega)$. Diminishing $|C|$, we diminish this open set until it contains the necessary interval (a', b'). The condition $\arg C + s\pi/2 \notin P$ prevents from the following situation: for some value of $|C|$ the intersection of the graphs of $\rho(\omega)$ and $\rho_g(\omega)$ is an arc of a sinusoid defined on an interval of length $> \delta/2$.

We end this section by two theorems the first of which was suggested

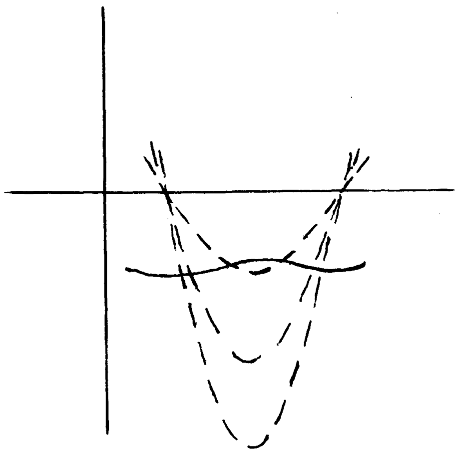

Fig. 4

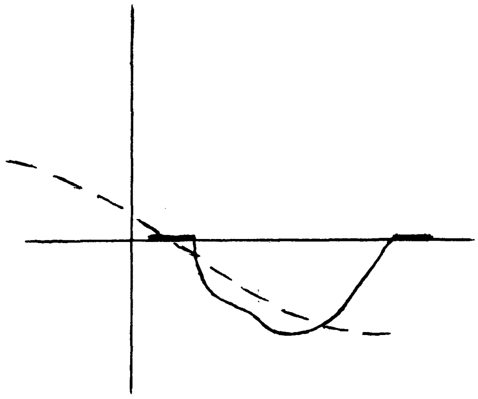

Fig. 5

by the author as a hypothesis and first proved by prof. J. Ecalle. We do not make use of this proof here, but we admit that the confirmation of the hypothesis stimulated the writing of this paper.

Theorem 4.6. *Let* $f \in B_\alpha$, $f\colon \{a \leq \arg z \leq b\} \to \boldsymbol{C}$ *and let* $f \in \tilde{A}_\beta^-$, $\beta > \alpha$ *for* $a' \leq \arg z \leq b'$, $a < a' \leq b' < b$, $b - a < \pi/\alpha$. *Then* $f \in A_\beta^-$ *for* $a \leq \arg z \leq b$ *and* $f \in \tilde{A}_\beta^-$ *for* $a < \arg z < b$.

Proof: Obviously, $f \in B_\beta$ for $a \leq \arg z \leq b$. It is clear that (with respect to β and not to α) we have $\rho_f(\omega) \leq 0$ for $\omega \in [a, a'] \cup [b', b]$ and $\rho_f(\omega) < 0$ for $\omega \in (a', b')$. Suppose that $\rho_f(\omega) = 0$ for $\omega \in [a, a'] \cup [b, b']$. Such a (continuous!) function cannot be subsinusoidal, see Fig. 5. The contradiction obtained with Theorems 4.3 and 4.4. shows that we must have $\rho_f < 0$ (with respect to β). Hence, $f \in A_\beta^-$ for $a \leq \arg z \leq b$ and $f \in \tilde{A}_\beta^-$ for $a < \arg z < b$.

Theorem 4.7. *Let* $f\colon S \to \boldsymbol{C}$ *be holomorphic inside and continuous and bounded on the closure of the sector* $S = \{a \leq \arg z \leq b, b - a < \pi/\alpha\}$. *Let* $f \in \tilde{A}_\alpha^-$ *for* $\arg z = \omega_0$ *with* $\rho_f(\omega_0) < 0$. *Then* $f \in \tilde{A}_\alpha^-$ *in any proper subsector of* S *(i.e. for* $a' \leq \arg z \leq b'$, *with arbitrary* $a < a' < b' < b$).

The proof is similar to the one of the previous theorem. We let the reader do it oneself.

Remark: It is possible to obtain results similar to the ones obtained in this section if one compares the growth rate of holomorphic functions in sectors not with e^{x^α}, but with $e^{x^\alpha \log^\beta x}$, $e^{x^\alpha \log^\beta x \log^\gamma \log x}$ etc.

5. Proof of Theorem 3.2

A) We say that a Stokes' multiplier is of level $\geq s$ if it has non-zero off-diagonal elements only on positions (j, k) corresponding to pairs (b_j, b_k) of level $\geq s$, see Section 2, 3). We consider solutions to system (3) in nice sectors. This means that all the Stokes' multipliers of level $q-1$ have minimal support.

Proposition 1. *The Stokes' multipliers of level $q-1$ preserve the relations in the differential field K_S.*

Proof: 1° Consider any two overlapping nice sectors S_1, S_2 and solutions W_1, W_2 with the same asymptotics (4) in them (for $\tau \to 0$) connected with each other by a Stokes' multiplier of level $q-1$: $W_2 = W_1 V$.

2° Present equality (7) in the form

$$\sum_j h_j(\tau) e^{d_j(1/\tau)} = 0 \tag{8}$$

where h_j are holomorphic functons in the sector S_1 with power asymptotics for $\tau \to 0$; the powers, in general, need not to be integer because of the presence of the factor $\exp(C \ln \tau)$, see (4); we can assume them to be positive. The polynomials $d_j(1/\tau)$ are of power $q-1$ and without a constant term.

3° Put $d_j(1/\tau) = d_{j,q-1}/\tau^{q-1} + \cdots + d_{j,1}/\tau$. Separate the terms with equal $d_{j,q-1}$ in (8):

$$\sum \tilde{h}_k(\tau) e^{d_{k,q-1}/\tau^{q-1}} = 0 \tag{9}$$

where $\tilde{h}_k \in B_{q-2}$ and all the $d_{k,q-1}$ are different. There exists a ray $l \subset S_1$ on which all the $d_{k,q-1}/\tau^{q-1}$ have different real parts. Let $d_{k_0,q-1}/\tau^{q-1}$ be the one with greatest real part. It is easy to see that then $\tilde{h}_{k_0}(\tau) = O(e^{-c/|\tau|^{q-1}})$ for some $c > 0$, otherwise equality (9) is impossible. Hence, $\tilde{h}_{k_0} \in \tilde{A}^-_{q-1}$ (this can be easily derived from Theorems 4.6 and 4.7). The sector S_1 is nice, i.e. of opening $> \pi/(q-1)$. Hence, $\tilde{h}_{k_0} \equiv 0$. Similarly we prove that $\tilde{h}_k \equiv 0$ for all k.

4° Apply the Stokes' multiplier V to the left hand side of equality (9) (remember that the functions $\tilde{h}_k$ are expressed by the elements of the matrices in the right hand side of (4)). This changes the functions $\tilde{h}_k$ to similar functons $\tilde{h}'_k \in B_{q-2}$ in S_2. In $S_1 \cap S_2$ $\tilde{h}_k$ and $\tilde{h}'_k$ differ by terms which are $O(e^{-c/|\tau|^{q-1}})$ for some $c > 0$; this follows from the fact that V is a Stokes' multiplier of level $q-1$ and, hence, the matrices $H(\tau)$ in the presentations (4) corresponding to W_1 and W_2 differ in $S_1 \cap S_2$ by terms of order $O(e^{-c/|\tau|^{q-1}})$. But then $\tilde{h}'_k \in \tilde{A}^-_{q-1}$ in $S_1 \cap S_2$. $\tilde{h}'_k \in B_{q-2}$ in S_2. According to Theorem 4.6 and Corollary 4.2, $\tilde{h}'_k \in \tilde{A}^-_{q-1}$ in S_2 and, hence, $\tilde{h}'_k \equiv 0$ (because S_2 is of opening $> \pi/(q-1)$). This proves the proposition.

B) Call a *good sector* of level $(q-s)$, $s=1, \dots, q-1$ any sector with vertex at 0 of opening $> \pi/(q-s)$ restricted by two Stokes' lines and not containing two different Stokes' lines corresponding to one and the same pair (b_j, b_k) of level $\le (q-s)$ inside itself. All nice sectors are good sectors of level $q-1$, the reverse may not be true.

Consider two good sectors of level $(q-s)$ — $\tilde{S}_1$, $\tilde{S}_2$. We assume that their intersection is restricted by two Stokes' lines —l_1, l_2—of level $(q-s)$ corresponding to one and the same pair of polynomials (b_j, b_k) which we call *the special pair*. Then $\tilde{S}_1 \cap \tilde{S}_2$ does not contain more than one Stokes' line corresponding to each pair (b_j, b_k) of level $\le (q-s)$. The coincidence of some of the Stokes' lines is of no importance for the further reasoning.

Remark: We assume that $\tilde{S}_1 \cup \tilde{S}_2$ contains no more than one Stokes' line different from l_1 and l_2 corresponding to each pair of polynomials (b_j, b_k) of level $q-s$ except for the special one. It is allowed, however, that the lines l_1, l_2 or one of them should correspond to two or more pairs of polynomials.

Consider coverings of $\tilde{S}_1$ and $\tilde{S}_2$ by good sectors of order $q-1$: let the sectors S', $S_1, \dots, S_\nu$ cover $\tilde{S}_1$ and let $S_1, \dots, S_\nu$, S'' cover $\tilde{S}_2$, each two adjacent sectors being overlapping, $S' \cup S_1 \cup \dots \cup S_\nu = \tilde{S}_1$, $S_1 \cup \dots \cup S_\nu \cup S'' = \tilde{S}_2$, see Fig. 6.

Proposition 2. *There exist solutions W', $W_1^j, \dots, W_\nu^j$, W'', $j=1, 2$ defined in S', $S_1, \dots, S_\nu$, S'' with one and the same asymptotics* (4) *in them. The matrices H (see* (4)) *of the solutions (W_k^j, W_{k+1}^j), $j=1, 2$; $k=1, \dots, \nu-1$ differ from each other by terms which are $O(e^{-c/|\tau|^{q-s+1}})$, $c>0$ in $S_k \cap S_{k+1}$ and so do the matrices H corresponding to (W', W_1^1) and (W_ν^2, W'') in $S' \cap S_1$ and $S_\nu \cap S''$ respectively. The matrices H of the solutions (W_k^1, W_k^2), $k=1, \dots, \nu$ differ from each other in S_k by terms which are $O(e^{-d/|\tau|^{q-s}})$, $d>0$.*

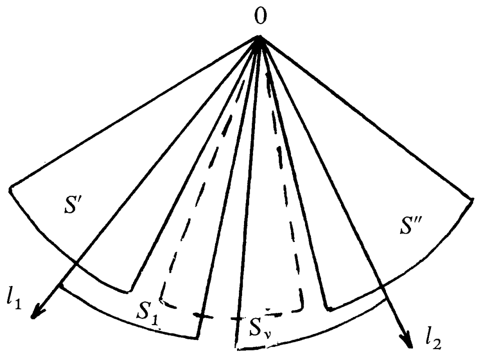

Fig. 6

Proof: 1° We take any solution with the asymptotics (4) in S' for W'. Let $W_1^1 = W'V$ where V is a Stokes' multiplier with minimal support. Put $V = V_0 V_1$ where V_0 (resp. V_1) is a Stokes' multiplier contaning non-zero off-diagonal elements only on positions corresponding to pairs of polynomials (b_j, b_k) of level $\geq q - s + 1$ (resp. of level $\leq q - s$). Such a decomposition is always possible. If $V_1 \neq I$, then we change (W', W_1^1) to $(W'V_1^{-1}, W_1^1)$. Thus the H-matrices of the new pair (W', W_1^1) differ in $S' \cap S_1$ by terms which are $O(e^{-c/|\tau|^{q-s+1}})$, $c > 0$.

2° In the same way we construct $W_2^1, W_3^1, \ldots, W_v^1$—constructing W_k^1 we may have to change $W_{k-1}^1, W_{k-2}^1, \ldots, W_1^1, W'$. After this we construct in a similar way W''; similar means that in the decomposition $V = V_0 V_1$ the off-diagonal non-zero elements of V_0 (resp. of V_1) correspond to pairs (b_j, b_k) of level $\geq q - s$ (resp. $\leq q - s - 1$). This is connected with the fact that the sector $\tilde{S}_1 \cup \tilde{S}_2$ contains two different Stokes' lines of level $q - s$ corresponding to the special pair.

3° Let $W_k^1 = W''V (1 \leq k \leq v)$. Put $V = V^0 V^1$ where V^0 contains non-zero off-diagonal elements corresponding only to pairs (b_j, b_k) of level $\geq q - s + 1$ and V^1—to the special pair. Put $W_k^2 = W''V^0$. We let the reader check oneself that the solutions W', W_k^1, W_k^2, $1 \leq k \leq v$, W'' constructed in this way satisfy the conditions of the proposition.

Corollary. *The solutions (W_k^1, W_k^2) are connected with each other by a Stokes' multiplier which has non-zero off-diagonal elements only on those positions which correspond to the special pair (b_j, b_k) of level $q - s$.*

This is a corollary not from Proposition 2 but from its proof—see 3°.

Proposition 3. *For every Stokes' multiplier V of level $q - s$ there exist the decompositions $V = V_0 V_1 = V_1' V_0$ where V_0 is a Stokes' multiplier of level $q - s$ not containing non-zero off-diagonal elements corresponding to pairs of polynomials of level $> q - s$ and V_1, V_1' are Stokes' multipliers of level $\geq q - s + 1$.*

Proof: A suitable ordering of the polynomials b_j brings V to the following block decomposition form (upper triangular, with units on the diagonal):

$$V = \begin{bmatrix} V^1 & K_1^2 & \ldots & K_1^r \\ 0 & V^2 & \ldots & K_2^r \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & V^r \end{bmatrix}$$

The off-diagonal elements of the blocks V^σ (the elements of the blocks K_μ^θ)

correspond to pairs (b_j, b_k) of level $q-s$ (of level $\geq q-s+1$). Hence, for suitable blocks $\tilde{K}^\theta_\mu$, $\bar{K}^\theta_\mu$ we have

$$V = \begin{bmatrix} V^1 & 0 & \dots & 0 \\ 0 & V^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V^r \end{bmatrix} \begin{bmatrix} I & \tilde{K}^2_1 & \dots & \tilde{K}^r_1 \\ 0 & I & \dots & \tilde{K}^r_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix}$$

$$= \begin{bmatrix} I & \bar{K}^2_1 & \dots & \bar{K}^r_1 \\ 0 & I & \dots & \bar{K}^r_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} \begin{bmatrix} V^1 & 0 & \dots & 0 \\ 0 & V^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V^r \end{bmatrix}$$

C) **Proposition 4.** *The Stokes' multipliers of level $\geq q-s$ with minimal support preserve the relations in the field K_S.*

Proof: 1° We prove Proposition 4 by induction; for $s=1$ it coincides with Proposition 1. It is easy to see that if the Stokes' multipliers of level $\geq q-s+1$ with minimal support preserve the relatons in the field K_S, then so do all Stokes' multipliers of level $\geq q-s+1$.

2° Relation (7) can be written in the form

$$\sum_j \tilde{h}_j(\tau) e^{d_j(1/\tau)} = 0 \tag{10}$$

where $d_j(1/\tau) = d_{j,q-1}/\tau^{q-1} + \cdots + d_{j,q-s}/\tau^{q-s}$ and $\tilde{h}_j$ are holomorphic functions belonging to the class B_{q-s-1}; the polynomials d_j are supposed different. To obtain equation (10) it is sufficient to express the left hand-side of equation (7) by the right hand-side of the representation (4) and then to separate the exponents of $1/\tau^{q-1}, \ldots, 1/\tau^{q-s}$ from the ones of $1/\tau^{q-s-1}, \ldots, 1/\tau$. It is clear that $\tilde{h}_j \in B_{q-s-1}$.

3° The functions $\tilde{h}_j$ can be expressed by different solutions to system (3), in different sectors (remember that the Stokes' multipliers of level $\geq q-s+1$ preserve the relations in the field K_S). Therefore we shall use the same indices for them as we use for the solutions W. For example, in S' and S_k the following equalities are true:

$$\sum_j \tilde{h}'_j(\tau) e^{d_j(1/\tau)} = 0 \quad \text{and} \quad \sum_j \tilde{h}^1_{j,k}(\tau) e^{d_j(1/\tau)} = 0$$

4° We make an additional non-restrictive assumption: the sectors $S' \backslash S_1$ and $S'' \backslash S_\nu$ contain no Stokes' lines of level $< q-1$ except l_1, l_2. We admit the situations $S_1 \subset S'$, $S_2 \subset S''$ provided that S_1, S_2 are good sectors. In this case the Stokes' multipliers $W^1_1(W')^{-1}$, $W^2_\nu(W'')^{-1}$ are trivial.

5° The functions $\tilde{h}'_j$, $\tilde{h}^1_{j,k}$, $k = 1, \ldots, v$ with j fixed in the intersections of the neighbouring sectors differ by terms which are $O(e^{-c/|\tau|^{q-s+1}})$. Choose a ray $l \subset S' \cap S_1$ on which one of the exponents—$e^{d_{j0}(1/\tau)}$—grows faster than the others for $\tau \to 0$. Then $\tilde{h}'_{j_0} = O(e^{-c'/|\tau|^{q-s}})$ for $\tau \in l$, otherwise equality (7) is impossible. But then $\tilde{h}'_{j_0} = O(e^{-c''/|\tau|^{q-s}})$ for $\tau \in S'$; this can easily be derived from Corollary 4.2 and Theorems 4.6 and 4.7. Hence, $\tilde{h}^1_{j_0,1} = \cdots = \tilde{h}^1_{j_0,v} = O(e^{-c'''/|\tau|^{q-s}})$.

6° There exists a change of coordinates in $\tilde{S}_1 \cup \tilde{S}_2$ $y = \tau\Phi(\tau)$, where $\Phi(\tau)$ increases or decreases for $\tau \to 0$ slowlier than any power of $|\tau|$, $\Phi \neq 0$ for $\tau \neq 0$ such that

1) the sectors S', $S_1, \ldots, S_v$, S'' are mapped onto curvilinear sectors S'^*, $S_1^*, \ldots, S_v^*$, S''^* with vertex at 0 such that the tangent lines to the sides of the sectors $S'^*, \ldots, S''^*$ at 0 coincide with the ones of the sectors $S', \ldots, S''$.

2) each one of the sectors $S'^*, \ldots$ contains a (true) subsector of smaller radius—$S'^{**}, \ldots$—and of opening $> \pi/(q-1)$ (with vertex at 0), each two adjacent subsectors being overlapping.

3) the function $\tilde{h}'_{j_0}(\tau(y))$ belongs to the class $\tilde{A}^-_\alpha \cap A_\alpha$ for some $\alpha \geq q - s$, $y \in S'^{**}$.

7° The existence of such a change of coordinates is proved in 9°–15°. Now we show that $\tilde{h}'_{j_0} \equiv 0$. Really, let $q - s + r > \alpha \geq q - s + r - 1$, $r \in N$. It follows from the construction of the solutions W', $W_1^1, \ldots, W_v^1$, W'', see the proof of Proposition 2, that there exists $\mu \in N$ such that the sector $S'^{**} \cup S_1^{**} \cup \cdots \cup S_\mu^{**}$ is of opening $> \pi/(q - s + r - 1)$ and all the differences $W' - W_1^1, \ldots, W_\mu^1 - W_{\mu-1}^1$ (defined on $S'^{**} \cap S_1^{**}, \ldots, S_{\mu-1}^{**} \cap S_\mu^{**}$) are $O(e^{-c^0/|y|^{q-s+r-\varepsilon}})$, $0 < \varepsilon < 1$, $c^0 > 0$. This follows from the fact that when we construct the solutions W', $W_1^1, \ldots, W_v^1$ we use the Stokes' multipliers with minimal support; for every pair of polynomials (b_j, b_k) of level $q - s_0 < q - 1$ the first Stokes' line in $\tilde{S}_1 \cap \tilde{S}_2$ corresponding to it (when we move from l_1 to l_2, see Fig. 6) does not introduce a difference between W_k^1 and W_{k+1}^1 or between W' and W_1^1 (which is $O(e^{-c^1/|\tau|^{q-s_0}})$) but only the second such line does. Hence, it follows from $\tilde{h}'_{j_0} \in \tilde{A}^-_\alpha \cap A_\alpha$ that $\tilde{h}^1_{j_0,1} \in \tilde{A}^-_\alpha \cap A_\alpha, \ldots, \tilde{h}^1_{j_0,\mu} \in \tilde{A}^-_\alpha \cap A_\alpha$ (there exists $\beta > \alpha$ such that $\tilde{h}'_{j_0} - \tilde{h}^1_{j_0,1} \in \tilde{A}^-_\beta, \ldots, \tilde{h}^1_{j_0,\mu-1} - \tilde{h}^1_{j_0,\mu} \in \tilde{A}^-_\beta$). It follows from Theorem 4.5 that $\tilde{h}'_{j_0} \equiv 0, \ldots, \tilde{h}^1_{j_0,\mu} \equiv 0$. Similarly we prove that $\tilde{h}'_j \equiv 0$ for all j. But then $\tilde{h}^1_{j,k} \equiv 0$, $k = 1, \ldots, v$ as well. This follows from the fact that the Stokes' multipliers of level $\geq q - s + 1$ preserve the relations in K_S.

8° In a similar way we prove that $\tilde{h}^2_{j,k} \equiv 0$, $k = 1, \ldots, v$, $\tilde{h}''_j \equiv 0$. It is essential in the proof that all the Stokes' lines of level $\leq q - s$ inside $\tilde{S}_1 \cap \tilde{S}_2$ are 'ignored', i.e. they don't participate with non-zero off-diagonal elements in the Stokes' multipliers connecting the solutions W', W_1^1, $W_1^2, \ldots, W_v^1$, W_v^2,

W''. The estimation for the rate of decreasing for $\tau \to 0$ of the difference $\tilde{h}^2_{j,k} - \tilde{h}^2_{j,k+1}$ is the same as for $\tilde{h}^1_{j,k} - \tilde{h}^1_{j,k+1}$, due to Proposition 3. Note that in the block decomposition of the Stokes' multiplier V, see the proof of Proposition 3, the blocks $\bar{K}^s_j$ and $\tilde{K}^s_j$ are or are not 0 simultaneously. These blocks correspond to one and the same pair of polynomials. Proposition 4 follows from the fact that $\tilde{h}^2_{j,k} \equiv 0$, $k = 1, \dots, v$, $\tilde{h}''_j \equiv 0$.

9° It remains to prove that the change of coordinates described in 7° really exists. Put $\alpha = \sup\{\delta \in \boldsymbol{R} \mid \exists C > 0, d > 0 \colon |\tilde{h}'_{j_0}(\tau)| \le Ce^{-d/|\tau|^\delta}, \tau \in l\}$. The restriction of $|\tilde{h}'_{j_0}|$ to l is a real-valued function h. We show in 10°–15° that it is possible to construct a holomorphic in $\tilde{S}_1 \cup \tilde{S}_2$ function $\psi(\tau)$ with the properties:

i) $\psi(\tau) \neq 0$ $\forall \tau \in \tilde{S}_1 \cup \tilde{S}_2 \backslash 0$

ii) $\lim_{\tau \to 0} |\tau|^s / |\psi(\tau)|$ is 0 for $s > 0$ and ∞ for $s < 0$

iii) $d(\tau\psi^{-1/\alpha}(\tau))/d\tau \neq 0$ in $\tilde{S}_1 \cup \tilde{S}_2 \backslash 0$

iv) Put $\gamma_1 = \limsup_{\tau \in l} h(\tau)/e^{-\psi(\tau)/|\tau|^\alpha}$. Then $0 < \gamma_1 < \infty$.

v) $\psi = |\psi|(1 + o(1))$ for $\tau \to 0$. In other words, $\arg\psi \to 0$ for $\tau \to 0$, $\tau \in \tilde{S}_1 \cup \tilde{S}_2$.

vi) the function $\tau\psi^{-1/\alpha}(\tau)$ is real for $\tau \in l$; without loss of generality we assume l to be the positive semi-axis. We also suppose the negative semi-axis not to belong to $\tilde{S}_1 \cup \tilde{S}_2$.

The necessary function $\Phi(\tau)$ is in fact $\psi^{-1/\alpha}$. Really, it follows from i), ii), iii) that the function $\tau\Phi(\tau)$ is a coordinate in $\tilde{S}_1 \cup \tilde{S}_2 \backslash 0$ (it might be necessary to diminish the radius of the sectors). It follows from iv) that $\rho_{\tilde{h}'_{j_0}}(0) = -\gamma_1$ with respect to α. Making use of Corollary 4.2, we conclude that in the new variable y we have $\tilde{h}'_{j_0} \in \tilde{A}^-_\alpha \cap A_\alpha$. It follows from v) that the sectors $S'^*, \dots, S''^*$ satisfy 1) and 2), see 6°.

10° Put $g(\tau) = -\log h(\tau)|\tau|^\alpha$, $g \colon \boldsymbol{R}^+ \to \boldsymbol{R}$; if $h(\tau_0) = 0$, then we put $g(\tau_0) = \infty$. The function $g(\tau)$ is bounded from below on any finite interval and $\lim_{\tau \to 0} |g| \cdot |\tau|^{-s} = \infty$ $\forall s > 0$. This follows from the definition of the number α. We prefer to construct the function ψ in the map $z = 1/\tau$. The image in this map of $\tilde{S}_1 \cup \tilde{S}_2$ is a sector S (without some compact set) of opening $< 2\pi$.

11° We can assume g to be positive (g cannot take negative values for arbitrarily large $|z|$, otherwise h will not be decreasing). Either $\inf_{z \ge z_0} g(z)$ increases slowlier or $\inf_{z \le z_0} g(z)$ decreases slowlier than any power of $|z_0|$ for $|z_0| \to \infty$; this follows from the definition of the number α. If $0 < \liminf_{|z| \to \infty} g(z) = \gamma_2 < \infty$, then we put $\psi = \gamma_2/2$. Let $\liminf_{|z| \to \infty} g(z) = 0$ (the case $\liminf_{|z| \to \infty} g(z) = \infty$ is treated in 14°). Then we put

$$\psi = \sum_{l=1}^{\infty} C_l/(z - a)^{k_l}$$

where $a < 0$, $(a \notin S)$, $C_l > 0$, $0 < k_l < 1/2^{(l+100)}$; the sequence $\{k_l\}$ is decreasing and tends to 0. The condition $k_l < 1/2^{(l+100)}$ implies that

$$\arg(C_l/(z-a)^{k_l}) \in [-\pi/4, \pi/4] \ \forall z \in S$$

Hence, all the partial sums and the sum of ψ (provided that it exists) have no zeros in S. The symbol $1/(z-a)^{k_l}$ denotes the sheat of a multivalued function taking real positive values for z real positive; k_l need not to be rational. If the numbers k_l tend to 0 very fast, then the arguments of the terms of the series tend to 0 very fast. We choose the constants C_l, k_l in such a way that $\psi(z)|_{\boldsymbol{R}} < g(z)$ and there exist infinitely many points z_1, z_2, $\ldots \to \infty$ such that $\psi(z_k) > g(z_k)/2$. This provides property iv); vi) is checked directly.

12° We prove here that the derivative of the function $z\psi^{1/\alpha}$ does not vanish in $S\backslash K$ for some compact set K; this implies iii). For this it is sufficient to prove that $D \equiv z d\psi/dz + \alpha\psi$ does not vanish there. But

$$D = \sum_{l=1}^{\infty} D_l\,, \qquad D_l = (\alpha C_l - C_l k_l)/(z-a)^{k_l} + aC_l k_l/(z-a)^{k_l+1}$$

If $k_l \ll \alpha$, $|a| k_l \ll 1$, $k_l < 1/2^{(100+l)}$, then for $|z| > 2|a|$ the module of the second term in D_l is much smaller than the module of the first one whose argument is close to 0 for l large; hence, for such k_l and z the arguments of all D_l belong to $[-\pi/4, \pi/4]$ and D does not vanish in $S\backslash K$. This proves iii); i) is proved in the same way. For $|z| \to \infty$ the argument of D is influenced stronger by the arguments of the D_l with larger l because they decrease slowlier But their argument is as closer to 0 as larger l is. Hence, for $|z| \to \infty$ $\arg \psi \to 0$. This proves v). Property ii) follows from the form of ψ.

13° Let's construct ψ itself. Let $z_0 > 1$. Put $\min_{z \le z_0} g(z) = m_0$. Put $\psi_s = \sum_{l=1}^{s} C_l/(z-a)^{k_l}$. Choose $0 < k_1 \ll \min(1/2^{101}, \alpha, 1/|a|)$, $C_1 > 0$ such that there exists $z_1 > z_0$: $g(z_1) < 2\psi_1(z_1) < 2g(z_1)$, $g(z) > \psi_1(z)$ for $z \ge 2|a|$, $\psi_1(2|a|) < m_0/2$; note that the functions $1/(z-a)^{k_l}$ are monotonously decreasing for $z \in \boldsymbol{R}$, $z > 2|a|$.

In the same way we construct ψ_2: $\exists z_2 > z_1 + 1$, $g(z_2) < 2\psi_2(z_2) < 2g(z_2)$, $g(z) > \psi_2(z) \forall z \ge 2|a|$, $\psi_2(2|a|) < m_0/2 + m_0/4$, $k_2 \ll \min(1/2^{102}, \alpha, 1/|a|)$, $C_2 > 0$ etc. The choice of the constants C_l, k_l is possible due to the slow rate of decreasing of $\inf_{z \le z_0} g(z)$ for $z_0 \to \infty$.

For $|z| = 2|a|$ we have $\psi_s = \sum_{l=1}^{s} C_l/(3a)^{k_l}$. We require that $|\psi_s|(2|a|) < (1-2^{-s})m_0$. It follows from here that the constants C_l are bounded and tend to 0. It is easy to prove the convergence of the series for ψ, for its derivative and for $d(z\psi^{1/\alpha})/dz$. This completes the construction of the function ψ in the case when $\inf_{z \le z_0} g(z)$ tends to 0.

14° Let now $\liminf_{z\to\infty} g(z) = \infty$. Put $G(z) = 1/g(z)$. Then G is bounded and $\sup_{|z|\geq z_0} G(z) \to 0$ slowlier than any power of $|z_0|$; this follows from the definition of the number α. Put $F = \psi^{-1}$,

$$F = \sum_{l=1}^{\infty} P_l\,, \qquad P_l = C_l/(z-a)^{k_l}$$

(the constants in the right hand-side have the same meaning as before). Put $F_s = P_1 + \cdots + P_s$. Let $\max_{z\geq 2|a|} G(z) = G(z_0)$. Choose $C_1 > 0$, $0 < k_1 \ll \min(\alpha, 1/\alpha, |a|, 1/|a|, 1/2^{101})$ such that $F_1(z_0) = G(z_0)$, $F_1(2|a|) \leq 2G(z_0)$, $\max_{z\geq 2|a|}(G - F_1) = (G - F_1)(z_1) \leq G(z_0)/2$, $z_1 > z_0 + 1$. Then similarly we choose C_2, k_2 such that $F_2(z_1) = G(z_1)$, $F_2(2|a|) \leq 2G(z_0) + 2(G - F_1)(z_1) \leq 3G(z_0)$, $\max_{z\geq 2|a|}(G - F_2) = (G - F_2)(z_2) \leq (G - F_1)(z_1)/2 \leq G(z_0)/4$, $z_2 > z_1 + 1$ etc. It is clear that $F_k > G$ for $z \leq z_k$, $F_k(z_s) < 4G(z_s)$, $s \leq k$; this follows from the fact that the functions $1/(z-a)^{k_l}$ are monotonous for $z > 2|a|$. But then $F(z_s) \leq 2G(z_s)$, $F(z) \geq G(z) \forall z \geq 2|a|$.

15° The properties i)–vi) for the function ψ can be checked similarly to the case when $\liminf_{z\leq z_0} g(z) = 0$, see 10°–13° and we prefer to let the reader check them oneself. This completes the proof of the proposition.

Proof of the Generalised Phragmen-Lindelöff Principle: There exists a change of coordinates in $S_1 \cup \cdots \cup S_k$ which maps the sectors S_j onto some curvilinear sectors S_j^* with vertex at 0 and which have the same tangent lines to their sides at 0 as the sectors S_k; the sectors S_j^* contain some true sectors of smaller radius S_j^{**} which are overlapping again. After this change of coordinates the functions f_j belong to the class $\tilde{A}_{\alpha_1}^- \cap A_{\alpha_1}$ for some $\alpha_1 > 0$ and satisfy all the other conditions of Theorem 4.5 in the sectors S_j^{**}; hence, they are 0. The construction of this change of coordinates is described in 9°–15° of the proof of Proposition 4.

Appendix. Proof of Theorems 4.3 and 4.4

The proofs of Theorems 4.3 and 4.4 follow from the lemmas formulated below. If the reader is reluctant to read the whole Appendix, he might read the lemmas and then pass onto the end of the Appendix where we explain how to derive the proofs of the theorems from the lemmas. For the sake of simplicity we put $\alpha = 1$ which leads to no loss of generality.

We use the notaton PC for the class of functions whose graphs consist of a finite number of arcs of the kind $y = k\cos(\omega + d)$, $d \in \boldsymbol{R}$. Hence, $PC^+ \subset PC$. The latter arcs are called *sinusoids*. Note that they *always* in this section have one and the same period -2π. We remind the notation $\|\cdot\| = \sup_{[a,b]}|\cdot|$. The reader should be careful in reading the proofs of the lemmas as we use the notation $(\cdot, \cdot)$ both to denote a point in $\boldsymbol{R}^2$ and an open interval in $\boldsymbol{R}$. The definition of a subsinusoidal function is given in Theorem 4.4.

Lemma 1. *Let the function* $f\colon [a, b] \to \boldsymbol{R}$, $b - a < \pi$ *be continuous and let there exist points* $a \leq \omega_0 < \omega_1 < \omega_2 \leq b$ *such that* $f(\omega_0) \leq 0$, $f(\omega_1) > 0$, $f(\omega_2) \leq 0$. *Then the function* f *does not belong to* $\overline{PC^+[a, b]}$.

Proof: 1° Suppose that for any $\delta > 0$ there exists $g \in PC^+$ such that $\| f + g \| < \delta$. Then we have $|f(\omega_1) - g(\omega_1)| < \delta$. Put

$$h = g(\omega_1) \cos(\omega - (\omega_0 + \omega_2)/2)/\cos(\omega_1 - (\omega_0 + \omega_2)/2)$$

Note that $h > 0$ for $\omega \in [\omega_0, \omega_2]$. Then there exists an arc of a sinusoid $\tilde{\alpha}$ which is a part of the graph of $g(\omega)$ such that $\tilde{\alpha} \cap \text{graph}(h) \ni (\omega_1, h(\omega_1))$. Either for $\omega = \omega_1 - 0$ or for $\omega = \omega_1 + 0$ the sinusoid $\tilde{\alpha}$ is above or on the graph of h (if the point $(\omega_1, h(\omega_1))$ is common for two such arcs of the graph of g, then at least one of them lies above or on the graph of h; this is easy to prove using the definition of the class PC^+, see Fig. 7).

2° Let $\tilde{\alpha}$ be above or on the graph of h for $\omega = \omega_1 + 0$. Then the first intersection point a^0 of the analytic continuation of $\tilde{\alpha}$ to the right with the ω-axis is to the right or coincides with the first one for the analytic continuation to the right of the graph of h (denoted by b^0). Hence, for $\omega \in (\omega_1, b]$ this continuation is strictly above the continuation of graph (h) or coincides with it. This follows from $\omega_2 - \omega_0 < \pi$. If $\tilde{\alpha}$ is defined for $\omega \in [\omega', \omega'']$, $\omega' \in [a, \omega_1]$, $\omega'' \in (\omega_1, b]$, then there exists another arc of sinusoid—$\tilde{\alpha}'$—defined for $\omega \in [\omega'', \omega''']$, $\omega''' \in (\omega'', b]$ which belongs to the graph of g, $\tilde{\alpha}' \cap \tilde{\alpha}'' = (\omega'', g(\omega''))$. The first intersection point of the analytic continuation of $\tilde{\alpha}'$ to the right with the ω-axis is strictly to the right of a_0 and, hence, to the right of b_0. This means that $\tilde{\alpha}'$ is strictly above the graph of h; this again follows from the definition of the class PC^+ and from $\omega_2 - \omega_0 < \pi$.

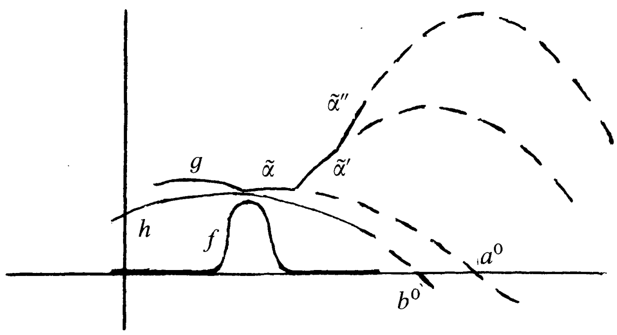

Fig. 7

3° If the graph of g contains more arcs of sinusoids to the right of $\tilde{\alpha}'$, then using the same arguments we show that these arcs lie above the graph of h. But

$$|h(\omega_2) - f(\omega_2)| \geq |h(\omega_2)| = |f(\omega_1)||\cos(\omega_2 - \omega_1)|/|\cos(\omega_1 - (\omega_0 + \omega_2)/2)|$$

which is greater than δ for δ sufficiently small. This proves the lemma.

Lemma 2. *Suppose that the function $f\colon [a, b] \to \boldsymbol{R}$, $b - a < \pi$ is continuous and there exist points $a \leq \omega_0 < \omega_1 < \omega_2 \leq b$ and a function $h = k \sin(\omega + d)$, $k, d \in \boldsymbol{R}$ such that $f(\omega_0) \leq h(\omega_0)$, $f(\omega_1) > h(\omega_1)$, $f(\omega_2) \leq h(\omega_2)$. Then the function f does not belong to $\overline{PC^+[a, b]}$.*

This lemma follows from Lemma 1 applied to the function $f - h$. We use the stability of $PC^+[a, b]$ under the addition of a sinusoid.

Lemma 3. *Let the function $f\colon [a, b] \to \boldsymbol{R}$, $b - a < \pi$ be continuous. Then f can be approximated by functions of the class PC.*

The proof consists in using the uniform continuity of f and the fact that through any two points not lying on a vertical line there passes exactly one sinusoid ($b - a < \pi$). We divide $[a, b]$ into intervals of length $\delta/2$, $0 < \delta < \pi/4$ such that the variation of f in each of them is $< \varepsilon$, $\varepsilon > 0$. The function f is bounded. Consider the set of functions $\tilde{G} = \{k \sin(\omega + d); k, d \in \boldsymbol{R}\}$. There exists $k_0 > 0$ such that if two points of the graph of f are connected by an arc of a function $g \in \tilde{G}$ and the difference between the ω-coordinates of the points is $< \delta$, then this arc is the graph of a monotonous function or of a function with $|k| \leq k_0$. The set $\tilde{G} \cap \{|k| \leq k_0\}$ is a set of functions with uniformly restricted derivatives on $[a, b]$. If the points (ω_1, y_1), (ω_2, y_2) belong to the graph of f, $|\omega_2 - \omega_1| < \delta$, then $|y_2 - y_1| < \varepsilon$ and one can easily estimate the variation of the arc of a sinusoid connecting these two points. We let the reader complete the proof oneself.

Lemma 4. *The class $\overline{PC^+[a, b]}$ contains only continuous functions.*

The proof is similar to the one of Theorem 4.2. If the function $f \in \overline{PC^+[a, b]}$ is not continuous, then there exists a function $h = k \sin(\omega + d)$ such that the function $f - g \in \overline{PC^+[a, b]}$ changes sign at least thrice on $[a, b]$, see Fig. 2 (remember that $b - a < \pi$). This contradicts Lemma 1.

We say that a function f admits a *special approximation* with functions of the class $PC[a, b]$ (or $PC^+[a, b]$) if for any $\delta > 0$ there exists $g \in PC[a, b]$ (or $g \in PC^+[a, b]$) such that $\|f - g\| < \delta$ and all the ends of arcs of sinusoids comprising the graph of the function g belong to the graph of f as well.

Lemma 5. *If a real function defined on $[a, b]$ is continuous and subsinusoidal, then it admits a special approximation with functions of the class $PC^+[a, b]$.*

Proof: 1° It follows from Lemma 3 that for any $0 < \delta < \pi/4$ there exists a function $g \in PC$ such that $\|f - g\| < \delta/4$. Denote by $a_1, \ldots, a_k$ the ends of the arcs of sinusoids comprising the graph of g. Consider one of the arcs $\tilde{\sigma}$, the graph of the restriction of g to $[a_j, a_{j+1}]$. There exist two possible cases:

1) $g(a_j) - f(a_j)$ *and* $g(a_{j+1}) - f(a_{j+1})$ have the same sign or one of them is 0

2) they have different signs.

2° In case 2) we replace the arc $\tilde{\sigma}$ by two arcs as follows: there exists $a' \in (a_j, a_{j+1})$ such that $g(a') = f(a')$. Let $g(a_j) - f(a_j) > 0$, $g(a_{j+1}) - f(a_{j+1}) < 0$ (the opposite case is treated similarly). We replace $\tilde{\sigma}$ by $\tilde{\sigma}' \cup \tilde{\sigma}''$ where $\tilde{\sigma}'$ is the restriction of $\tilde{\sigma}$ to $[a_j, a_{j+1}]$ and $\tilde{\sigma}''$ is the arc of a sinusoid defined on $[a', a_{j+1}]$ connecting the points $(a', g(a'))$ and $(a_{j+1}, f(a_{j+1}))$. On $[a_j, a']$ the difference $|f - g|$ does not change. For $\omega \in [a', a_{j+1}]$ the difference $|(\tilde{\sigma}'' - \tilde{\sigma})(\omega)|$ is greatest for $\omega = a_{j+1}$ and, hence, less than $\delta/4$ because $\tilde{\sigma}''(a_{j+1}) = f(a_{j+1})$. Thus for $\omega \in [a', a_{j+1}]$

$$|f - g| \leq |f - \tilde{\sigma}| + |\tilde{\sigma} - \tilde{\sigma}''| < \delta/4 + \delta/4 = \delta/2 < \delta$$

3° Consider case 1). Let the ends of $\tilde{\sigma}$ be the points (a_j, b_j), (a_{j+1}, b_{j+1}). Consider the arc of a sinusoid $\tilde{\sigma}'''$ connecting the points (a_j, b_j) and $(a_{j+1}, f(a_{j+1}))$. Then $\tilde{\sigma}''' \cap \tilde{\sigma}$ is the point (a_j, b_j). The difference $|b - b'''|$ where $(a, b) \in \tilde{\sigma}$, $(a, b''') \in \tilde{\sigma}'''$ is greatest for $a = a_{j+1}$. Hence, for any $a \in [a_j, a_{j+1}]$ and $(a, b) \in \tilde{\sigma}$, $(a, b''') \in \tilde{\sigma}'''$ we have

$$|b''' - f(a)| \leq |b''' - b| + |b - f(a)| < \delta/4 + \delta/4 = \delta/2$$

Let now in a similar way $\tilde{\sigma}^0$ be the arc of a sinusoid connecting the points $(a_j, f(a_j))$ and $(a_{j+1}, f(a_{j+1}))$. The difference $|b^0 - b'''|$ where $(a, b^0) \in \tilde{\sigma}^0$, $(a, b''') \in \tilde{\sigma}'''$ is greatest for $a = a_{j+1}$ and we know that $|f(a_j) - b_j| < \delta/4$. Hence,

$$|b^0 - f(a)| \leq |b^0 - b'''| + |b''' - f(a)| < \delta/4 + \delta/2 < \delta$$

Thus, uniting cases 1) and 2), we are able for any function $f \in \overline{PC^+[a, b]}$ and for any $\delta > 0$ to construct a function $g \in PC$ such that $\|f - g\| < \delta$.

4° It remains to show that $g \in PC^+[a, b]$. If this is not true, then some pair of arcs AC, BD of the graph of g have analytic continuations which look like it is shown on Fig. 8. The arc AB is below these continuations and the graph of f is below or on AB. This means that the approximation is not special. The lemma is proved.

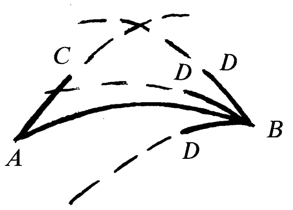

Fig. 8.

Call a <u>special sequence</u> (approximating the function g) a sequence of functions $g_j \in \overline{PC^+[a,b]}$ which are special approximations of g and such that if D_j denotes the set of the ends of arcs of sinusoids building the graph of g_j, then $D_j \subset D_{j+1}$, $j = 1, 2, \ldots$.

Lemma 6. *Any function $f \in \overline{PC^+[a,b]}$ can be approximated by a special sequence.*

Proof: We replace the sequence $\{g_j\}$ (in case that it is not special) by a sequence $\{g_j^0\}$ where the set of the ends of arcs of sinusoids comprising the graph of g_j^0 is $D_1 \cup D_2 \cup \cdots \cup D_j$. g is subsinusoidal (according to Lemma 2). Hence, the newly built sequence (which is special) provides a not worse approximation than the initial one. We leave the details for the reader.

Lemma 7. *For every function $g \in \overline{PC^+[a,b]}$ there exists a function $f \in A_\alpha$ such that $g = \rho_f(\omega)$.*

Proof: 1° Let $\alpha = 1$ (this assumption is non-restrictive). Put $S = \{a \leq \arg z \leq b, b - a < \pi\}$. It is sufficient to prove the lemma for $f \in \tilde{A}_\alpha^- \cap A_\alpha$ which can always be achieved by multiplying f by a decreasing in the sector exponent. This means that the function g takes negative values only.

2° Consider a decreasing sequence of numbers $k_j > 1$, $k_j \to 1$. Choose a decreasing sequence of numbers $\varepsilon_j > 0$, $\varepsilon_j \to 0$ such that $|k_{j+1} - k_j||g(\omega)| > \varepsilon_j$ $\forall \omega \in [a,b]$. It follows from Lemma 5 that for each of the functions $k_j g(\omega)$ there exists a function $g_j(\omega) \in PC^+[a,b]$ such that $|(g_j - k_j g)(\omega)| < \varepsilon_j/2$. This implies that

1) $\forall \omega \in [a,b]$ the sequence $\{g_j(\omega)\}$ is strictly monotonously increasing

2) $\lim_{j\to\infty} g_j(\omega) = g(\omega)$ $\forall \omega \in [a,b]$, the convergence being uniform in $[a,b]$

3) there exist functions $F_j = 1/q_j \sum_{q=1}^{q_j} p_{j,q} e^{c_{j,q}z}$, $q_j \in N$, $p_{j,q} = \min(1, 1/|c_{j,q}|)$ such that $g_j = \rho_{F_j}(\omega)$ (see the lemma in Section 4 and the definition of the class $PC^+[a,b]$). The exponents $e^{c_{j,q}z}$ are decreasing in S.

3° The necessary function f is constructed as a sum of the kind

$$f = \sum_{j=1}^{\infty} C_j F_j$$

where $0 < C_j \leq 1/2^{j-1}$. It follows from the definition of the functions F_j that this series and the series of its derivatives are uniformly convergent in S. Hence, this series is convergent to a holomorphic in the sector S function.

4° Put $C_1 = 1$. Further by K_j we denote a circle of radius r_j centered at 0. Each F_j is a finite sum of exponents and $\forall\omega \in [a, b]$ $\rho_{F_{j+1}}(\omega) > \rho_{F_j}(\omega)$. Hence, there exists K_1 and a constant $D_1 > 0$ such that $\forall\omega \in [a, b]$ $\exists z^0$: $\arg z^0 = \omega$, $z^0 \in K_1 \cap S$, $|F_1(z^0)| > D_1 e^{\rho_{F_1}(\omega)|z^0|}$. Choose $0 < C_2 \leq 1/2$ such that $\forall\omega \in [a, b]$ we would have $|C_2F_2(z^0)| < |F_1(z^0)|/4$. There exist $D_2 > 0$ and K_2, $r_2 > r_1 + 1$ such that $\forall\omega \in [a, b] \exists z^1$: $\arg z^1 = \omega$, $z^1 \in (K_2 \backslash K_1) \cap S$, $|(F_1 + C_2F_2)(z^1)| > D_2 e^{\rho_{F_1+C_2F_2}(\omega)|z^1|} \equiv D_2 e^{\rho_{F_2}(\omega)|z^1|} > D_1 e^{\rho_{F_1}(\omega)|z^1|}$.

5° Choose $0 < C_3 \leq 1/4$ such that $\forall\omega \in [a, b]$ we would have $|C_3F_3(z^0)| < |F_1(z^0)|/8$, $|C_3F_3(z^1)| < |(F_1 + C_2F_2)(z^1)|/4$. There exist $D_3 > 0$, K_3, $r_3 > r_2 + 1$ such that $\forall\omega \in [a, b] \exists z^2 \in (K_3 \backslash K_2) \cap S$, $\arg z^2 = \omega$,

$$\begin{aligned} |(F_1 + C_2F_2 + C_3F_3)(z^2)| &> D_3 e^{\rho_{F_1+C_2F_2+C_3F_3}(\omega)|z^2|} \\ &> D_2 e^{\rho_{F_1+C_2F_2}(\omega)|z^2|} \\ &> D_1 e^{\rho_{F_1}(\omega)|z^2|} \qquad \text{etc} \end{aligned}$$

Finally we would have that $\forall\omega \in [a, b] \exists \{z^j\}$, $|z^j| \to \infty$ such that $|f(z^j) > (D_k/2) e^{\rho_{F_1+\cdots+C_kF_k}(\omega)|z^j|}$ for $j \in N \cup 0$, $k \leq j + 1$. This means that $\forall\omega \in [a, b] \rho_f(\omega) > \rho_{F_k}(\omega)$, $k = 1, 2, \ldots$. On the other hand-side it is checked directly that

$$|f| \leq \left| \sum_{s=1}^{\infty} C_j/q_j \sum_{s=1}^{q_j} e^{c_{j,s}z} \right| \leq \sum_{j=1}^{\infty} (C_j/q_j) q_j e^{g(\omega)} = \text{const } e^{g(\omega)}$$

i.e. $\rho_f(\omega) \leq g(\omega)$. Hence, $\rho_f(\omega) \equiv g(\omega)$.

Lemma 8. *Let* $f \in \overline{PC^+[a, b]}$. *Then* f *is Lipschitz on every interval* $[a', b']$, $a < a' < b' < b$.

We shall not prove the lemma in full detal. Let the lemma be not true. Then there exists $F \in A_\alpha$ such that $f = \rho_F(\omega)$ and the function f is not Lipschitz at some point. Then there exists a functon $G = e^{kz}$ such that the function ρ_{FG} provides a contradiction with Theorem 4.1. Examples of the possible graphs of f and ρ_G are shown on Fig. 9.

Proof of Theorem 4.3 *and* 4.4: It follows from Theorems 4.1 and 4.2 that any function of the class

$$U = \{f \colon [a, b] \to \boldsymbol{R}, b - a < \pi/\alpha | f = \rho_F(\omega), F \in A_\sigma\}$$

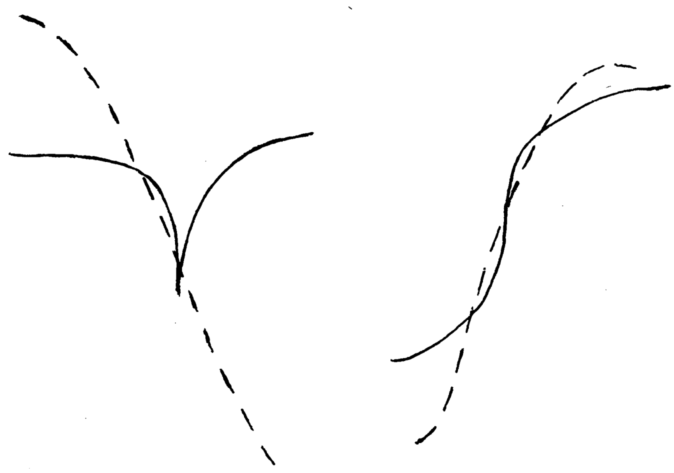

Fig. 9

is subsinusoidal and continuous. It follows from Lemmas 1 and 2 that any function $f \in \overline{PC^+[a, b]}$ is subsinusoidal and from Lemma 4—that it is continuous. It follows from Lemmas 3 and 5 that any subsinusoidal continuous function defined on $[a, b]$ belongs to $\overline{PC^+[a, b]}$. This together with Lemma 8 proves Theorem 4.4. Theorem 4.3 follows now from Lemma 7.

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